

Topological Excitations and their Contribution to Quantum Criticality in 2 + 1 D Antiferromagnets

Zaira Nazario[†] and David I. Santiago^{†,*}

[†] Department of Physics, Stanford University, Stanford, California 94305

^{*} Gravity Probe B Relativity Mission, Stanford, California 94305

(Dated: February 6, 2008)

It has been proposed that there are new degrees of freedom intrinsic to quantum critical points that contribute to quantum critical physics. We study 2+1 D antiferromagnets in order to explore possible new quantum critical physics arising from nontrivial topological effects. We show that skyrmion excitations are stable at criticality and have nonzero probability at arbitrarily low temperatures. To include quantum critical skyrmion effects, we find a class of exact solutions composed of skyrmion and antiskyrmion superpositions, which we call topolons. We include the topolons in the partition function and renormalize by integrating out small size topolons and short wavelength spin waves. We obtain a correlation length critical exponent $\nu = 0.9297$ and anomalous dimension $\eta = 0.3381$.

PACS numbers: 75.10.-b, 75.40.Cx, 75.40.Gb, 75.40.-s

There have been recently interesting suggestions that there are intrinsic degrees of freedom at quantum critical points and that these intrinsic critical excitations are different from the elementary excitations the quantum critical point separates^{1,2}. In order to explore a path toward this possible new physics at quantum critical points, we study the approach to the quantum critical point from the Néel ordered phase of 2+1 D antiferromagnets. A physics that is usually not included in the studies of criticality is the effects of the nontrivial topology of the $O(3)$ group on the quantum critical physics. In the present article we study this topology and include its effects on the critical physics. We also calculate its contribution to critical exponents. In the end we assess how close or how far we are from finding the effective field theory of the critical point deriving from its intrinsic critical excitations.

I. 2 + 1 D O(3) NONLINEAR SIGMA MODEL IN THE STEREOGRAPHIC PROJECTION LANGUAGE

We now turn to 2+1 D antiferromagnets whose effective field theory is given by the nonlinear σ model augmented by Berry phase terms^{3,4}:

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}\vec{n} \delta(\vec{n}^2 - 1) e^{-S} \\ S &= S_B + \frac{\rho_s}{2} \int_0^\beta d\tau \int d^2\vec{x} \left[(\partial_{\vec{x}}\vec{n})^2 + \frac{1}{c^2} (\partial_\tau\vec{n})^2 \right] \quad (1) \\ &= S_B + \frac{\Lambda}{2g} \int_0^\beta d(c\tau) \int d^2\vec{x} \left[(\partial_{\vec{x}}\vec{n})^2 + \frac{1}{c^2} (\partial_\tau\vec{n})^2 \right] \end{aligned}$$

where a is the lattice constant, $\Lambda = 1/a$, S is the microscopic spin with \hbar included (not to be confused with the Euclidean action), and J is the microscopic spin exchange. $\rho_s \equiv JS^2/\hbar$ is the spin stiffness, $c = 2\sqrt{2}JSa$ is the spin-wave velocity, and the dimensionless microscopic coupling constant is $g = 2\sqrt{2}/S$. These values are obtained for the case of nearest neighbor Heisenberg interactions only. For this case, in order to move the system from a Néel ordered phase ($g \ll 1$) to a disordered or quantum paramagnetic phase ($g \gg 1$), we tune the microscopic spin from large to small values. In real life one does not have this option as the microscopic spin is fixed. Moreover, the smallest available spin $S = 1/2$ is not enough to place the system in the quantum paramagnetic phase⁵. In order to quantum disorder the system in real life, one would need to add frustrating next nearest neighbor interactions J' , frustrating ring exchange interactions K , or other longer range but still short range interactions that compete with the nearest neighbor Néel order interaction. In this case, the dimensionless coupling g becomes a function of the ratios of the different interactions $g = g(J/J', J/K, \dots)$ and by tuning the competing interactions, one can take the system from the Néel ordered to the quantum paramagnetic phase.

The Berry phase term is the sum of the areas swept by the vectors $\vec{n}_i(\tau)$ on the surface of a unit sphere at each lattice site as they evolve in Euclidean time³. The Berry phase terms are zero when the Néel magnetization is continuous, i.e. Néel ordered phase and critical point, as the contributions from neighboring lattice sites cancel^{3,6,7}. Since we will concentrate on critical properties as approached from the Néel ordered side, the Berry phase vanishes.

A particular parametrization of the nonlinear sigma model that will prove specially convenient when we shortly move to study the topological structure of the model is obtained by mapping the staggered magnetization \vec{n} to the complex variable w via the stereographic

⁰ Z.N. is now at the Max Planck Institute for the Physics of Complex Systems in Dresden, Germany. D.I.S. is now at the Instituut-Lorentz at Leiden University in Leiden, The Netherlands.

mapping⁸

$$n^1 + in^2 = \frac{2w}{|w|^2 + 1}, \quad n^3 = \frac{1 - |w|^2}{1 + |w|^2}, \quad w = \frac{n^1 + in^2}{1 + n^3}. \quad (2)$$

More details of the stereographic projection are given in Appendix A. In terms of w the nonlinear sigma model action is

$$\begin{aligned} S &= \frac{2\Lambda}{g_\Lambda} \int d^3x \frac{\partial^\mu w \partial_\mu w^*}{(1 + |w|^2)^2} \\ &= \frac{2\Lambda}{g_\Lambda} \int d^3x \frac{\partial_0 w \partial_0 w^* + 2\partial_z w \partial_{z^*} w^* + 2\partial_{z^*} w \partial_z w^*}{(1 + |w|^2)^2} \end{aligned} \quad (3)$$

where $\partial_0 = \partial/\partial(c\tau)$. We have renamed g as g_Λ because this is the microscopic coupling constant. In order to study the phases of the model and the quantum critical point, we will perform renormalization group studies^{5,9,10,11,12,13}. When we integrate degrees of freedom, the coupling constants will become renormalized, but retain the same form since the nonlinear sigma model is a renormalizable field theory^{11,12,14}. Thus the renormalized theory takes the form

$$S = \frac{2\mu}{g_\mu} \int d^3x \frac{\partial^\alpha w \partial_\alpha w^*}{(1 + |w|^2)^2} \quad (4)$$

with μ is the sliding renormalization scale¹⁵ and g_μ the renormalized coupling constant. We have suppressed field renormalization factors.

A. Traditional Goldstone Renormalization in the Stereographic Projection Language

In order to gain experience with stereographic projections, to show that the results are equivalent to more used approaches^{5,10,11,12,13,16} and for later use we now perform a one loop renormalization expansion of the nonlinear sigma model.

In terms of w , the partition function is

$$\mathcal{Z} = \prod_{\tau, \vec{x}} \frac{\mathcal{D}w(\tau, \vec{x}) \mathcal{D}w(\tau, \vec{x})^*}{(1 + |w(\tau, \vec{x})|^2)^2} e^{-S} \quad (5)$$

where the Euclidean action S is given by (3). A tricky point in the partition function is the nontrivial measure arising from the nonlinearity of the sigma model and from the Jacobian of the transformation between the \vec{n} variables and the stereographic projection w :

$$\prod_{\tau, \vec{x}} \frac{1}{(1 + |w|^2)^2} = \exp \left[-2 \sum_{\tau, \vec{x}} \ln(1 + |w|^2) \right] \quad (6)$$

These products, or the sum in the exponentials, are not well defined in the continuum limit. How this term is

treated depends on your regularization method. For example, in dimensional regularization this term is just ignored with the price of some infrared divergences that must be treated carefully. On the other hand, direct passage to the continuum limit yields

$$\begin{aligned} \exp \left[-2 \sum_{\tau, \vec{x}} \ln(1 + |w|^2) \right] &= \\ \exp \left[-2\delta^3(0) \int d^3x \ln(1 + |w|^2) \right]. \end{aligned} \quad (7)$$

The delta functions are ill defined as they are infinite. This can be dealt with in at least two equivalent ways, either by introducing a lattice cutoff in real space or a momentum cutoff in momentum space. We thus have

$$\begin{aligned} \exp \left[-2 \sum_{\tau, \vec{x}} \ln(1 + |w|^2) \right] &= \\ \exp \left[-\frac{2}{a^3} \int d^3x \ln(1 + |w|^2) \right] \end{aligned} \quad (8)$$

where space integrals are cutoff at the short distance a . Then the partition function can be written as

$$\mathcal{Z} = \int \prod_{\tau, \vec{x}} \mathcal{D}w(\tau, \vec{x}) \mathcal{D}w^*(\tau, \vec{x}) e^{-S_{eff}} \quad (9)$$

where the effective action is the nonlinear sigma model action augmented by the terms obtained from the nontrivial measure:

$$S_{eff} = \frac{2}{a^3} \int d^3x \ln(1 + |w|^2) + S. \quad (10)$$

We are interested in the Néel ordered phase and in the critical point, but this last will be studied as it is approached from the Néel ordered phase. So g_Λ will not be too large and, at least in the Néel ordered phase, a perturbative expansion in g_Λ is possible. Renormalization group resummations of this expansion yield approximations to the critical theory. The lowest order terms in g_Λ will be given by the approximation

$$\begin{aligned} S_{eff} \simeq \frac{2}{a^3} \int d^3x |w|^2 + \frac{2\Lambda}{g_\Lambda} \int d^3x \partial^\mu w \partial_\mu w^* \\ - \frac{4\Lambda}{g_\Lambda} \int d^3x |w|^2 \partial^\mu w \partial_\mu w^*. \end{aligned} \quad (11)$$

We now work in momentum space. The effective action is then

$$\begin{aligned}
S_{eff} = & \frac{2V^2}{a^3} \int \frac{d^3k}{(2\pi)^3} |w(k)|^2 + \frac{2V^2\Lambda}{g_\Lambda} \int \frac{d^3k}{(2\pi)^3} k^2 |w(k)|^2 \\
& - \frac{4V^4\Lambda}{g_\Lambda} \int \frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^9} w(k_1) w^*(k_2) k_3 \cdot (k_1 - k_2 + k_3) w(k_3) w^*(k_1 - k_2 + k_3)
\end{aligned} \tag{12}$$

where $k = (\omega, \vec{k})$ and V is the volume where the system lies. We will slightly change our definition of the cut-off in order to absorb some angular factors. Hence we renormalize the momentum sphere to

$$V_3^\Lambda = \frac{4\pi}{3} \Lambda^3 = \frac{1}{a^3}. \tag{13}$$

We'll renormalize the theory via momentum shell integration. In order for this action to be well defined, i.e. finite, it is cutoff at large momentum Λ and we'll integrate the degrees of freedom from Λ to a smaller cutoff μ . The bare action is

$$S_0 = 2V^2 \frac{\Lambda}{g_\Lambda} \int \frac{d^3k}{(2\pi)^3} k^2 |w(k)|^2 \tag{14}$$

which leads to the bare momentum space Green's function or propagator

$$G_0 = \frac{(2\pi)^3}{2V^2} \frac{g_\Lambda}{\Lambda} \frac{1}{k^2}. \tag{15}$$

The interaction term is given by

$$\begin{aligned}
S_I = & 2V^2 V_3^\Lambda \int \frac{d^3k}{(2\pi)^3} |w(k)|^2 \\
& - 4V^4 \frac{\Lambda}{g_\Lambda} \int \left\{ \frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^9} w(k_1) w^*(k_2) \right. \\
& \times k_3 \cdot (k_1 - k_2 + k_3) w(k_3) w^*(k_1 - k_2 + k_3) \} .
\end{aligned} \tag{16}$$

We know integrate the high momentum degrees of freedom to obtain the effective theory at scale μ . If the momentum shell is thin enough, a perturbative expansion is

justified and to lowest order in the interaction we have

$$\mathcal{Z} \simeq \int_\Lambda \mathcal{D}w \mathcal{D}w^* (1 - S_I) e^{-S_0}. \tag{17}$$

After integrating the high energy degrees of freedom ($\mu < k < \Lambda$) we obtain

$$\mathcal{Z} \simeq \int_\mu \mathcal{D}w \mathcal{D}w^* e^{-S_{eff}^\mu}. \tag{18}$$

where the effective action at scale μ is given by

$$S_{eff}^\mu = S_0 + \langle S_I \rangle \tag{19}$$

and the average of S_I is taken over the large momentum degrees of freedom. The noninteracting action S_0 is the one obtained from S_0 at scale Λ by integrating out the degrees of freedom with momenta between μ and Λ and the irrelevant constant term are thrown out as they only modify the overall normalization. Now we evaluate the expectation value of S_I term by term. We first obtain

$$\begin{aligned}
\langle S_I^{(1)} \rangle = & 2V^2 V_3^\Lambda \int \frac{d^3k}{(2\pi)^3} \langle |w(k)|^2 \rangle \\
= & 2V^2 V_3^\Lambda \int_{|k|<\mu} \frac{d^3k}{(2\pi)^3} |w(k)|^2 + C
\end{aligned} \tag{20}$$

C is a constant; and

$$\begin{aligned}
\langle S_I^{(2)} \rangle = & -4V^4 \frac{\Lambda}{g_\Lambda} \int \left\{ \frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^9} \langle w(k_1) w^*(k_2) k_3 \cdot (k_1 - k_2 + k_3) w(k_3) w^*(k_1 - k_2 + k_3) \rangle \right\} \\
& - 2V^2 (V_3^\Lambda - V_3^\mu) \int_{|k|<\mu} \frac{d^3k}{(2\pi)^3} |w(k)|^2 - \frac{V^2}{\pi^2} [\Lambda - \mu] \int_{|k|<\mu} \frac{d^3k}{(2\pi)^3} k^2 |w(k)|^2 + C
\end{aligned} \tag{21}$$

where the constant terms will be neglected as they only provide a change of overall normalization of the partition

function. Putting everything together we obtain

$$\begin{aligned}
S_{eff}^\mu &= 2V^2 V_3^\mu \int_{|k|<\mu} \frac{d^3 k}{(2\pi)^3} |w(k)|^2 + 2V^2 \frac{\Lambda}{g_\Lambda} \left\{ 1 - \frac{g_\Lambda}{2\pi^2} \left[1 - \frac{\mu}{\Lambda} \right] \right\} \int_{|k|<\mu} \frac{d^3 k}{(2\pi)^3} k^2 |w(k)|^2 \\
&- 4V^4 \frac{\Lambda}{g_\Lambda} \int_{|k_i|<\mu} \frac{d^3 k_1 d^3 k_2 d^3 k_3}{(2\pi)^9} \{ w(k_1) w^*(k_2) k_3 \cdot (k_1 - k_2 + k_3) w(k_3) w^*(k_1 - k_2 + k_3) \} \\
&\simeq 2V^2 V_3^\mu \int_{|k|<\mu} \frac{d^3 k}{(2\pi)^3} |w(k)|^2 + 2V^2 \frac{\mu}{g_\mu} \int_{|k|<\mu} \frac{d^3 k}{(2\pi)^3} k^2 |w(k)|^2 \\
&- 4V^4 \frac{\mu}{g_\mu} \int_{|k_i|<\mu} \frac{d^3 k_1 d^3 k_2 d^3 k_3}{(2\pi)^9} \{ w(k_1) w^*(k_2) k_3 \cdot (k_1 - k_2 + k_3) w(k_3) w^*(k_1 - k_2 + k_3) \}.
\end{aligned} \tag{22}$$

We thus see that, to lowest order, the action renormalizes^{5,10,11,12}, i.e. goes into itself when fast degrees of freedom are integrated out. This result can be proved to be true to all orders by means of the $O(3)$ Ward identity of the $O(3)$ nonlinear sigma model¹⁴.

The renormalized coupling constant at scale μ is seen to be

$$g_\mu = \left(\frac{\mu}{\Lambda} \right) \frac{g_\Lambda}{1 - \frac{g_\Lambda}{2\pi^2} \left[1 - \frac{\mu}{\Lambda} \right]} . \tag{23}$$

This expression exemplifies quite a bit of the physics of the renormalization group and the nonlinear sigma model. We first see that the coupling constant gets renormalized to different values at different scales. From this expression we see at least two fixed point values of the coupling constant in the sense that for these two values the coupling constant is the same at all momentum scales. One is $g_\Lambda = g_{IR}^N = 0$ and the other is at $g_\Lambda = g_{UV}^c = 2\pi^2$ as found long ago¹⁰. g_{IR}^N corresponds to the Néel ordered or Goldstone phase and it is an infrared fixed point which corresponds to a stable phase of matter. It is an infrared fixed point because if $0 < g_\Lambda < g_{UV}^c$, $g_\mu \rightarrow g_{IR}^N = 0$ as $\mu \rightarrow 0$. That is, as the energy scale is lowered, the theory approaches the Néel ordered behavior.

The critical point where Néel order is lost corresponds to g_{UV}^c . That this is a critical point follows since it is an infrared unstable fixed point. If g_Λ is arbitrarily close but not exactly g_{UV}^c , it deviates from g_{UV}^c as $\mu \rightarrow 0$. This is easily seen from our formula if g_Λ is chosen to be a value infinitesimally smaller than g_{UV}^c . More importantly, the critical point is an ultraviolet fixed point because if $0 < g_\Lambda < g_{UV}^c$, $g_\mu \rightarrow g_{UV}^c$ as $\mu \rightarrow \infty$. All critical points are ultraviolet fixed points. Therefore, the critical properties can be studied from the Néel ordered phase by studying their high momentum or high energy behavior. Finally, the nonlinear sigma model has another infrared fixed point at $g_\Lambda = \infty = g_{IR}^P$, which corresponds to the paramagnetic phase. This fixed point cannot be accessed by expanding about the Néel ordered phase as it is not adiabatically continuable to the Néel ordered phase. In order to access this paramagnetic fixed point, one needs to perform a strong coupling expansion.

The spin stiffness of the nonlinear sigma model is pro-

portional to the inverse coupling constant

$$\rho_s \propto \frac{\mu}{g_\mu} . \tag{24}$$

Classically, $g_\mu/\mu = g_\Lambda/\Lambda$ is a constant and does not get renormalized. The spin stiffness only vanishes when the bare coupling constant g_Λ becomes infinite. When fluctuation effects are included, g_μ becomes renormalized according to

$$\rho_s \propto \frac{\Lambda}{g_\Lambda} \left[1 - \frac{g_\Lambda}{2\pi^2} \left(1 - \frac{\mu}{\Lambda} \right) \right] \tag{25}$$

If we tune g_Λ to the critical value where Néel order is lost, g_{UV}^c , the spin stiffness is

$$\rho_s \propto \frac{\mu}{g_{UV}^c} . \tag{26}$$

We thus see that at the quantum critical point, $g_\Lambda = g_{UV}^c = 2\pi^2$, the spin stiffness vanishes at arbitrarily low energy scales $\mu \rightarrow 0$.

The beta function is calculated from our expression for the renormalized coupling constant g as a function of μ , (23), to be

$$\beta(g) = \mu \frac{\partial g}{\partial \mu} \Big|_{\Lambda=\mu} = g - \frac{g^2}{2\pi^2} . \tag{27}$$

The fixed points are given by the zeros of the β function, which are easily seen to occur at $g = 0$ and $g = 2\pi^2$ as found before. The first term, which is positive, arises because in more than two space-time dimensions the theory is not truly scale invariant but becomes so at fixed points. In more than two space-time dimensions, the presence of the negative term makes possible the existence of a critical point because at certain value of the coupling constant, the negative or asymptotic freedom term cancels the positive scaling term of the β function.

Since our renormalization of the nonlinear sigma model was carried out to one loop, we did not obtain the renormalization of the fields. The situation is not as bad as it seems as it is possible to obtain a field renormalization by a different one loop calculation. More important

than the renormalization of the w fields, we are interested in the renormalization Z of the staggered magnetization. Because of the $O(3)$ invariance, the Goldstone fields $\Pi_1 = n_1$ and $\Pi_2 = n_2$, and the ordering direction or $\sigma = n_3$ component are renormalized by the same factor $\sqrt{Z} = \sqrt{1 + \delta} \simeq 1 + \delta/2$. We will fix the renormalized magnetization σ to one in the Néel ordered phase. On the other hand, the bare magnetization is related to the renormalized one via $\sigma_0 = \sqrt{Z} \sigma \simeq (1 + \delta/2) \sigma$. In order to obtain the renormalized magnetization we calculate

$$\begin{aligned}\sigma_0 &= \langle n_3 \rangle = \left\langle \frac{1 - |w(\tau, \vec{x})|^2}{1 + |w(\tau, \vec{k})|^2} \right\rangle \simeq 1 - 2\langle |w(\tau, \vec{x})|^2 \rangle \\ \sigma_0 &= \sqrt{Z} \simeq 1 + \frac{\delta}{2}\end{aligned}\quad (28)$$

where the average value is obtained by integrating the fast degrees of freedom between scales μ and Λ in order to obtain the magnetization at scale μ . Therefore we obtain the counterterm

$$\begin{aligned}\delta &= -4\langle |w(\tau, \vec{x})|^2 \rangle = -4V^2 \int_{\mu < |k| < \Lambda} \frac{d^3 k}{(2\pi)^3} G_0(k) \\ &= -\frac{g_\Lambda}{\pi^2} \left[1 - \frac{\mu}{\Lambda} \right].\end{aligned}\quad (29)$$

N -point Green's functions of the nonlinear sigma model satisfy the Callan-Symanzik (CS) equation^{11,12,15,17,18}

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + \frac{N}{2} \gamma(g) \right] G^{(N)}(p, g, \mu) = 0 \quad (30)$$

where the anomalous dimension $\gamma(g)$ is given by

$$\gamma(g) = \mu \frac{\partial \ln Z}{\partial \mu} \Big|_{\mu=\Lambda} \simeq \mu \frac{\partial \delta}{\partial \mu} \Big|_{\mu=\Lambda} = \frac{g}{\pi^2} \quad (31)$$

We are particularly interested in the critical properties of antiferromagnets and hence the nonlinear sigma model. When tuned to criticality, we obtain that exactly at the UV fixed point or critical point, $g_\Lambda = g_{UV}^c = 2\pi^2$, the Green's function has the form

$$G^{(2)}(p, g_{UV}^c, \mu) = \frac{1}{p^2} h \left(\frac{p^2}{\mu^2} \right) \quad (32)$$

which means that if we exchange the μ derivative for a p derivative the CS equation becomes

$$\left[p \frac{\partial}{\partial p} - \gamma(g_{UV}^c) \right] h \left(\frac{p^2}{\mu^2} \right) = 0. \quad (33)$$

This integrates immediately to

$$h \left(\frac{p^2}{\mu^2} \right) = A' \left(\frac{\mu^2}{p^2} \right)^{-\gamma/2} \quad (34)$$

where A' is an arbitrary integration constant. The two point critical function is then

$$\begin{aligned}G^{(2)}(p, g_{UV}^c, \mu) &= \frac{A'}{\mu^{\gamma(g_{UV}^c)/2}} \left(\frac{1}{p^2} \right)^{1-\gamma(g_{UV}^c)/2} \\ &\equiv \frac{A}{p^{2-\eta}}.\end{aligned}\quad (35)$$

From equation (31), we immediately find $\gamma(g_{UV}^c) = 2$ in agreement with previous one loop results^{10,11}. This is obviously wrong as the propagator loses all momentum dependence. The reason for this nonsense is that higher order corrections are quite large and must be included. More careful approximations give less large and more sensible values^{5,14,16}. We will evaluate below, for the first time, the corrections coming from topological effects in order to get a more accurate evaluation of the anomalous exponents.

Better than tuning the system exactly to criticality it is more realistic to study the Green's function in the Néel ordered phase. The two point Green's function in the Néel ordered phase takes the form

$$\begin{aligned}G^{(2)}(p, g, \mu) &= \frac{1}{p^2} h \left(\frac{\mu^2}{\Lambda^2} \right) = \frac{1}{p^2} h(g_\mu) \\ &= \frac{1}{p^2} h \left(g, \frac{\mu^2}{p^2} \right).\end{aligned}\quad (36)$$

Because h is a function of μ^2/Λ^2 only through the coupling constant. Hence the CS equation can be integrated immediately to give

$$\begin{aligned}G^{(2)}(p, g, \mu) &= \frac{A}{p^2} \exp \left[- \int_{g_p}^{g_\mu} dg' \frac{\gamma(g')}{\beta(g')} \right] \\ &= \frac{A}{p^2} \left(1 - \frac{g_\mu}{g_{UV}^c} \left[1 - \frac{p}{\mu} \right] \right)^2\end{aligned}\quad (37)$$

We now see that the Green's function interpolates between the Goldstones behavior at long wavelengths, $p \rightarrow 0$

$$G^{(2)}(p, g, \mu) = \frac{A}{p^2} \left(1 - \frac{g_\mu}{g_{UV}^c} \right)^2 \quad (38)$$

and critical behavior with anomalous exponent $\eta = 2$ at short distances, $p \rightarrow \infty$.

$$G^{(2)}(p, g, \mu) = \frac{A}{p^2} \left[\frac{g_\mu}{g_{UV}^c} \right]^2 \frac{p^2}{\mu^2} = \frac{A}{\mu^2} \left[\frac{g_\mu}{g_{UV}^c} \right]^2 \quad (39)$$

Despite the large incorrect value of γ , the qualitative conclusions obtained are true. There is a nonzero anomalous exponent at the critical point and the Green's function satisfy the Callan-Symanzik equation. Moreover, the conclusions drawn from the Callan-Symanzik equation, i.e. the scaling behavior, becomes quantitatively correct at small energy scales⁵. In fact, the quantitative agreement between RG studies of the nonlinear sigma model and neutron scattering results in the cuprate high T_c superconductors demonstrated that these are Néel ordered at low temperatures when sufficiently underdoped⁵.

For completeness and perspective, we finish this section by extracting the information that follows from the CS equation (30). In order to integrate this equation, we introduce the magnetization $\sigma = \sigma(g)$ obtained by

integrating the equation

$$\begin{aligned} 0 &= \left(\beta(g) \frac{\partial}{\partial g} + \frac{1}{2} \gamma(g) \right) \sigma(g) \\ \sigma(g_\mu) &= B \exp \left\{ - \int^{g_\mu} \frac{\gamma(g)}{2\beta(g)} dg \right\} \\ &\simeq M \left[1 - \frac{g_\mu}{2\pi^2} \right] = M (1 - g_\mu/g_{UV}^c) \end{aligned} \quad (40)$$

where M is an arbitrary integration constant. Before continuing with the analysis of the CS equation, we briefly digress to discuss the properties of the coupling-dependent renormalized magnetization $\sigma(g_\mu)$. For $g_\mu < 2\pi^2 = g_{UV}^c$, i.e. in the Néel ordered phase, the renormalized magnetization $\sigma(g_\mu)$ is a nonzero constant. We see that as the coupling constant is tuned to the critical value g_{UV}^c , the magnetization goes to zero with the critical exponent $\beta = 1^1$. Since this is a one loop result, the exponent is not accurate but the fact that the magnetization goes to zero at criticality is true.

Let us go back to the analysis of the CS equation. In order to simplify it, we define the correlation length via the equation

$$\begin{aligned} 0 &= \left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right) \xi_J(\mu, g) \\ \xi_J(g_\mu, \mu) &= \frac{1}{\mu} \exp \left\{ \int^{g_\mu} \frac{dg'}{\beta(g')} \right\} \\ &= \frac{1}{\mu} \left[\frac{g_\mu/g_{UV}^c}{1 - g_\mu/g_{UV}^c} \right] \end{aligned} \quad (41)$$

This correlation length is the Josephson correlation length $\xi_J^{5,16}$ which determines the crossover from short distance critical behavior to long distance Goldstone behavior. Just as with the magnetization, this determination of the correlation length is not accurate enough because it is a one loop result. We see that the correlation length diverges with the exponent $\nu = 1$.

II. TOPOLOGICAL EXCITATIONS OF NÉEL ORDERED 2+1 D ANTIFERROMAGNETS

2+1 D antiferromagnets and their effective description via the $O(3)$ nonlinear sigma model have a classical “ground state” or lowest energy state with Néel order corresponding to a constant magnetization. The equations of motion that follow from the action have approximate time dependent solutions, corresponding to Goldstone spin wave excitations. The equations of motion, in 2+1 D only, also have exact static solitonic solutions of finite energy²⁰. Digressing for a moment from antiferromagnets, since the 1970’s, it has been known that

when systems have exact classical, time independent solutions which are stable against quantum fluctuations, these solutions are quantum particle excitations of the system¹⁹. The nonlinear sigma model, in 2+1 D only, possesses time independent solutions which are of a topological nature^{8,20} and have finite energy. These excitations are disordered at finite length scales but relax into the Néel state far away

$$\lim_{|\vec{x}| \rightarrow \infty} \vec{n} = (0, 0, 1), \quad \lim_{|\vec{x}| \rightarrow \infty} w = 0. \quad (42)$$

They consist in the order parameter rotating a number of times as one moves from infinity toward a fixed but arbitrary position in the plane. Since two dimensional space can be thought of as an infinite 2 dimensional sphere, the excitations fall in homotopy classes of a 2D sphere into a 2D sphere: $S^2 \rightarrow S^2$. The topological excitations are thus defined by the number of times they map the 2D sphere into itself. They are characterized by the Jacobian

$$q = \frac{1}{8\pi} \int d^2x \epsilon^{ij} \vec{n} \cdot \partial_i \vec{n} \times \partial_j \vec{n} \quad (43)$$

or in terms of the stereographic variable

$$\begin{aligned} q &= \frac{i}{2\pi} \epsilon^{ij} \int d^2x \frac{\partial_i w \partial_j w^*}{(1 + |w|^2)^2} \\ &= \frac{1}{\pi} \int d^2x \frac{\partial_z w \partial_{z^*} w^* - \partial_{z^*} w \partial_z w^*}{(1 + |w|^2)^2}. \end{aligned} \quad (44)$$

The number q will be an integer measuring how many times the n -sphere gets mapped into the infinite 2D sphere corresponding to the plane where the spins live. If we define the space-time current

$$J^\mu = \frac{1}{8\pi} \epsilon^{\mu\nu\sigma} \vec{n} \cdot \partial_\nu \vec{n} \times \partial_\sigma \vec{n} = \frac{i}{2\pi} \epsilon^{\mu\nu\sigma} \frac{\partial_\nu w \partial_\sigma w^*}{(1 + |w|^2)^2}, \quad (45)$$

it is easily seen that it is conserved $\partial_\mu J^\mu = 0$ and that the charge associated with it is our topological charge

$$q = \int d^2x J^0. \quad (46)$$

Thus q is a conserved quantum number. These topological field configurations were originally discovered by Skyrme²¹ and are called skyrmions. The conserved charge is the skyrmion number.

Since the skyrmions are time independent, their energy is given by

$$E_s = \frac{4\Lambda}{g_\Lambda} \int d^2x \frac{\partial_z w \partial_{z^*} w^* + \partial_{z^*} w \partial_z w^*}{(1 + |w|^2)^2}. \quad (47)$$

From this expression and the one for the charge q , it is easily seen^{8,20} that $E \geq 4\pi|q|\Lambda/g_\Lambda$. We see that we can construct skyrmions with $q > 0$ by imposing the condition

$$\partial_{z^*} w = 0 \quad (48)$$

¹ This is not the β function, but this exponent is called β for convention. We hope this does not cause confusion.

that is w is a function of z only. The magnetization, \vec{n} or w , is an analytic function of z almost everywhere. The worst singularities it can have are poles. The skyrmions will have a location given by the positions of the poles or of the zeros of w . Far away from its position, the field configuration will relax back to the original Néel order. Therefore we have the boundary condition $w(\infty) = 0$, which implies

$$w = \lambda^q \prod_{i=1}^q \frac{1}{z - a_i} \quad (49)$$

which can easily be checked to have charge q and energy

$$E_s(q) = 4\pi \frac{\Lambda}{g_\Lambda} q = 4\pi \rho_s q. \quad (50)$$

λ^q is the arbitrary size and phase of the configuration and a_i are the positions of the skyrmions that constitute the multiskyrmion configuration. The energy is independent of the size and phase due to the conformal invariance of the configuration. We remark that since the multiskyrmions energy is the sum of individual skyrmion energies, the skyrmions do not interact among themselves⁸. Similarly, the multiantiskyrmion configuration can be shown to be

$$w = (\lambda^*)^q \prod_{i=1}^q \frac{1}{z^* - a_i^*} \quad (51)$$

with charge $-q$ and energy $4\pi\Lambda q/g_\Lambda$. Skyrmions and antiskyrmions do interact as shown in Appendix A. In that Appendix we collect these results on skyrmions and provide further developments.

We now investigate whether skyrmions and anti-skyrmion configurations are relevant at the quantum critical point. As mentioned above, their classical energy is $4\pi\Lambda/g_\Lambda$, which is independent of the size of the skyrmion λ . On the other hand, in real physical systems there are quantum and thermal fluctuations. These renormalize the effective coupling constant of the nonlinear sigma model and make it scale dependent. To one loop order the renormalized coupling constant is

$$g_\mu = \frac{\mu}{\Lambda} \frac{g_\Lambda}{1 - (g_\Lambda/2\pi^2)(1 - \mu/\Lambda)}. \quad (52)$$

Since the skyrmion has an effective size λ , spin waves of wavelength smaller than λ renormalize the energy of the skyrmion via the coupling constant renormalization. The energy E of the skyrmion at the scale set by its size $\mu = 1/\lambda$ is now

$$E_s = \frac{4\pi\mu}{g_\mu} \Big|_{\mu=1/\lambda} = \frac{4\pi\Lambda}{g_\Lambda} \left[1 - \frac{g_\Lambda}{2\pi^2} \left(1 - \frac{1}{\lambda\Lambda} \right) \right]. \quad (53)$$

We see that the skyrmion energy now depends on its size through the renormalization effects and thus the conformal invariance of the configuration is broken. This is

the well known phenomenon of broken scale invariance in renormalizable theories due to the scale dependence of the coupling constant^{18,22}. We see that if we tune the system to criticality, $g_\Lambda = 2\pi^2$, the energy of a skyrmion of size λ is $E_s = 2/(\pi\lambda)$ and the energy of excitation for skyrmions of arbitrarily large size is zero and hence degenerate with the ground state. The quantum critical point thus seem to be associated with skyrmion gap collapse. In order to see if skyrmion fluctuations are indeed relevant to the critical point, let's be a little bit more careful and explicit.

If the system is at temperature $T = 1/\beta$, this temperature sets the size of the skyrmion to be the thermal wavelength $\lambda = \beta$. The skyrmion Euclidean action is then

$$S_s = \frac{4\pi\beta}{\beta g_{1/\beta}} = \frac{4\pi\beta\Lambda}{g_\Lambda} \left[1 - \frac{g_\Lambda}{2\pi^2} \left(1 - \frac{1}{\beta\Lambda} \right) \right] \quad (54)$$

The probability for skyrmion creation is given by $P \propto e^{-S_s}$. Let us see how this probability behaves at low temperatures. When in the Néel ordered phase, $g_\Lambda < g_c = 2\pi^2$, the skyrmion Euclidean action diverges as $T \rightarrow 0$. Therefore, the probability for skyrmion contributions is suppressed exponentially at low temperatures, vanishing at zero temperature. *Skyrmions are gapped and hence irrelevant to low temperature physics in the Néel ordered phase.*

At the quantum critical point $g_\Lambda = g_c = 2\pi^2$, the skyrmion Euclidean action is

$$S_s = \frac{2}{\pi}. \quad (55)$$

This action is finite and constant at all temperatures and in particular, it will have a nonzero limit as the temperature goes to 0: the skyrmion probability is nonzero and constant at arbitrarily low temperatures and zero temperature. *Hence there are skyrmion excitations at criticality at arbitrarily low energies and temperatures, including at zero temperature. Therefore skyrmion excitations contribute to quantum critical physics.*

III. TOPOLOGICALLY NONTRIVIAL CONFIGURATIONS WITH ZERO SKYRMION NUMBER

We have seen that skyrmions are relevant at criticality as the critical point is associated with skyrmion gap collapse and they have a nonzero probability to be excited at arbitrarily low temperature at criticality. On the other hand, skyrmions have nonzero conserved topological number while the ground state has zero skyrmion number. Absent any external sources that can couple directly to skyrmion number, they will always be created in equal numbers of skyrmions and antiskyrmions. Therefore, in order to study the effect of skyrmions and anti-skyrmions we need to include configurations with equal

number of skyrmions and antiskyrmions in the partition function.

We need to find nontrivial solutions of the nonlinear sigma model equations of motion with zero skyrmion number that corresponds to superpositions of equal number of skyrmions and antiskyrmions. The classical equations of motion which follow by stationarity of the classical action are

$$\begin{aligned}\square w &= \frac{2w^*}{1+|w|^2} \partial^\mu w \partial_\mu w \\ \partial_0^2 w - 4\partial_z \partial_{z^*} w &= \frac{2w^*}{1+|w|^2} \{(\partial_0 w)^2 - 4\partial_z w \partial_{z^*} w\}\end{aligned}\quad (56)$$

We are interested in time independent and finite energy (or nonzero probability) solutions. For time independent solutions, the structure of the equations suggests a solution of the form

$$w = e^{i\varphi} \tan \left[f(z) + (f(z))^* + \frac{\theta}{2} \right] \quad (57)$$

where φ and θ are arbitrary, constant angles and $f(z)$ is an arbitrary function of z only and not of z^* , which is analytic in z almost everywhere (just as with skyrmion and antiskyrmion solutions, the function can have poles but no worse singularities). These solutions are topologically trivial since they have zero skyrmion number. On the other hand, as we shall see explicitly below, the finite energy solutions will correspond to arbitrary superpositions of equal number of skyrmions and antiskyrmions with $q = \pm n$. Since, despite being topologically trivial, these solutions will be composed of topologically non-trivial configurations (skyrmions and antiskyrmions), we dub them topolons. We hope this name does not create confusion.

The first steps we need to check that the topolon solution written above is a solution is to compute the derivatives. We start with the simplest ones. Since our solutions are time independent, we immediately obtain

$$\partial_0 w = 0, \quad \partial_0^2 w = 0. \quad (58)$$

The next derivatives to calculate are

$$\begin{aligned}\partial_z w &= e^{i\varphi} f'(z) \sec^2 \left[f(z) + (f(z))^* + \frac{\theta}{2} \right] \\ \partial_{z^*} w &= e^{i\varphi} (f'(z))^* \sec^2 \left[f(z) + (f(z))^* + \frac{\theta}{2} \right] \\ \partial_z \partial_{z^*} w &= 2e^{i\varphi} |f'(z)|^2 \sec^2 \left[f(z) + (f(z))^* + \frac{\theta}{2} \right] \\ &\quad \times \tan \left[f(z) + (f(z))^* + \frac{\theta}{2} \right].\end{aligned}\quad (59)$$

Using these derivatives, we see that the left hand side of the equation of motion (56) is

$$\begin{aligned}\text{LHS} &= -8e^{i\varphi} |f'(z)|^2 \sec^2 \left[f(z) + (f(z))^* + \frac{\theta}{2} \right] \\ &\quad \times \tan \left[f(z) + (f(z))^* + \frac{\theta}{2} \right].\end{aligned}\quad (60)$$

and the right hand side is

$$\begin{aligned}\text{RHS} &= \frac{2e^{-i\varphi} \tan \left[f(z) + (f(z))^* + \frac{\theta}{2} \right]}{1 + \tan^2 \left[f(z) + (f(z))^* + \frac{\theta}{2} \right]} \\ &\quad \left\{ -4e^{2i\varphi} |f'(z)|^2 \sec^4 \left[f(z) + (f(z))^* + \frac{\theta}{2} \right] \right\} \\ &= -8e^{i\varphi} |f'(z)|^2 \sec^2 \left[f(z) + (f(z))^* + \frac{\theta}{2} \right] \\ &\quad \times \tan \left[f(z) + (f(z))^* + \frac{\theta}{2} \right].\end{aligned}\quad (61)$$

The left hand side and the right hand side are identical and therefore the topolon solution written above is indeed an exact solution.

The energy of this solution is easily calculated to be

$$E = \frac{8\Lambda}{g_\Lambda} \int d^2x \left| \frac{df(z)}{dz} \right|^2. \quad (62)$$

In order for the solution to have finite energy, $f(z)$ must decay faster than $1/z^\xi$ as $z \rightarrow \infty$, with $\xi > 0$. But since $f(z)$ is analytic except for poles or multipoles, $f(z)$ decays at least as fast as $1/z$ as $z \rightarrow \infty$. Therefore, a general form that has finite energy is given by

$$f_n(z) = \lambda^n \prod_{i=1}^n \frac{1}{z - a_i} \quad (63)$$

with w_n given as above with $f(z) = f_n(z)$. This is the general n -topolon. Two parameters the n -topolon depends on are the two orientation angles. In general, skyrmions depend on two orientation angles which are given by the orientation of the Néel order the skyrmion relaxes on far away. These are usually not counted when counting the parameters of a skyrmion as they are usually fixed to constant values by the boundary conditions. Besides these two angles, a skyrmion depends on an arbitrary complex parameter λ and n complex positions when we have an n -skyrmion. The n -topolon depends also on an arbitrary complex parameter λ and n complex positions. Besides the arbitrariness of these parameters (the same arbitrariness as for an n -skyrmion), there is no other arbitrariness to the topolon as its form is dictated by solving the equations of motion.

Given that the argument of the topolon is precisely the sum of an n -skyrmion with an n -antiskyrmion, it is clear that a topolon is in general a superposition of skyrmions and antiskyrmions. The antiskyrmions are at the same positions as the skyrmions, i.e. the argument of the topolon has paired skyrmions and antiskyrmions at the same position. If this was not the case, we would not have a solution of the equations of motion. This is perhaps not surprising because if we take a look at the skyrmion-antiskyrmion interaction we obtained in Appendix A, it has a minimum when the skyrmion and antiskyrmion are at the same position, that is, the relative distance is zero.

IV. RENORMALIZATION OF THE NONLINEAR SIGMA MODEL INCLUDING GOLDSSTONE AND TOPOLONS

We have just found time independent solutions to the equations of motion given by

$$w_t^{(n)} = e^{i\varphi} \tan \left[\prod_{i=1}^n \frac{\lambda}{z - a_i} + \prod_{i=1}^n \frac{\lambda^*}{z^* - a_i^*} + \frac{\theta}{2} \right] \quad (64)$$

which corresponds to superpositions of equal number of skyrmions and antiskyrmions. These solutions contribute to the partition function for the antiferromagnet, which is then given by

$$\mathcal{Z} = \sum_{n=0}^{\infty} \mathcal{Z}_n \quad (65)$$

where

$$\mathcal{Z}_0 = \int \frac{\mathcal{D}\nu \mathcal{D}\nu^*}{(1 + |\nu|^2)^2} e^{-S[\nu]} \quad (66)$$

is the usual partition function for the nonlinear sigma model with no topolons and only the spin wave like fields ν , and S is the Euclidean action for the nonlinear sigma model in terms of the stereographic projection variable $w = \nu$. We also have that

$$\begin{aligned} \mathcal{Z}_{n \neq 0} &= \int \frac{\mathcal{D}\nu \mathcal{D}\nu^*}{(1 + |w_t^{(n)} + \nu|^2)^2} \frac{d\Omega}{4\pi} \frac{d\lambda}{1/\Lambda} \\ &\times \prod_{i=1}^n \frac{d^2 a_i}{A} e^{-S[w_t^{(n)} + \nu]}. \end{aligned} \quad (67)$$

The $\mathcal{Z}_{n \neq 0}$ is the path integral with the n topolons with spin waves ν . Besides integrating over the spin wave configurations, we must integrate over the topolon parameters: its size λ normalized to the lattice spacing $1/\Lambda$, the positions a_i of the topolon constituents skyrmions and antiskyrmions, normalized to the area A of the system, and over the solid angle of the topolon orientation normalized to 4π .

The next step to evaluate the partition function, or to evaluate expectation values from it, would be to expand the topolon action in a semiclassical expansion about the spin waves. This process can be simplified by realizing that the same physics follows if all the constituents skyrmions are placed in the same position. This result is not obvious so we will review the main points of how it comes about. When the semiclassical expansion is performed, as will be described below, we obtain a sum of terms in the partition function, each of which is weighted by $e^{-\beta E_n}$ where E_n is the n -topolon energy. This energy is given by

$$E_n = \frac{8\pi\Lambda n}{g_\Lambda} \left[\frac{\lambda}{R(a_1, a_2, \dots, a_n)} \right]^{2n} \quad (68)$$

where R is a distance that depends on the particular positions of the skyrmions and antiskyrmions making up the arguments of the n -topolon. All terms in the semiclassical expansion, after being multiplied by the weight factor, need to be integrated over the solid angle of the orientation of the topolons ($d\Omega/4\pi$) and over the positions of the n skyrmions and antiskyrmions that constitute the argument of the n -topolon ($d^2 a_1 d^2 a_2 \dots, d^2 a_n / (\text{area})^n$). The only dependence on the a_i 's in the weighting factor appears through $R(a_1, \dots, a_n)$. We also need to integrate over λ . Integration over solid angles of each of the terms of the semiclassical expansion yields a constant independent of a_i 's. That is, the angular average over powers of $w_t^{(n)}$ is independent of the positions a_i 's.

We make a change of variables and interchange the variable a_1 with the variable R . When we do this, we are left with integrals with volume element

$$\frac{\partial a_1}{\partial R} d\lambda dR \prod_{i=2}^n da_i. \quad (69)$$

We then perform a change of variables in the $\lambda - R$ plane:

$$\lambda' = \frac{\lambda}{R}, \quad R' = R. \quad (70)$$

The volume element of our integral becomes

$$R' \frac{\partial a_1}{\partial R'} d\lambda' dR' \prod_{i=2}^n da_i. \quad (71)$$

Now the weighting factor $e^{-\beta E_n}$ depends on λ' only and is independent of R' and the a_i 's. Hence the integrations over R' and the a_i 's can be performed yielding an irrelevant constant factor which can be absorbed in the normalization or dropped. We are left with an integration over λ' and a weighting factor which is equal to the one obtained with the n skyrmions and antiskyrmions that constitute the n -topolon placed in the same position

$$w_t^{(n)} = e^{i\varphi} \tan \left[\left(\frac{\lambda}{z - a} \right)^n + \left(\frac{\lambda^*}{z^* - a^*} \right)^n + \frac{\theta}{2} \right] \quad (72)$$

since the Euclidean action of such a configuration is given by

$$\begin{aligned} S_t &= \beta E_n \\ &= \frac{4\beta\mu}{g_\mu} \int d^2x \frac{\partial_z w_t^{(n)} \partial_{z^*} w_t^{(n)*} + \partial_{z^*} w_t^{(n)} \partial_z w_t^{(n)*}}{(1 + |w_t^{(n)}|^2)^2} \\ &= \frac{8\beta\mu}{g_\mu} n^2 \lambda^{2n} \int d^2x \frac{1}{|z|^{2(n+1)}} = \frac{8\pi\beta\mu}{g_\mu} n (\lambda\Lambda)^{2n}. \end{aligned} \quad (73)$$

For the topolon the scale is set by the size of the topolon, so we choose $\mu = 1/\lambda$. In the case where λ is set by the temperature ($\beta = \lambda$), we have

$$\begin{aligned} S_t &= \beta E_n = \int_0^\lambda d\tau \frac{8\pi}{\lambda g_{1/\lambda}} n (\lambda\Lambda)^{2n} \\ &\simeq \frac{8\pi\lambda\Lambda}{g_\Lambda} n (\lambda\Lambda)^{2n}. \end{aligned} \quad (74)$$

Without loss of generality, from now on we only consider the topolon with all skyrmions constituting $f(z)$ placed at the same position a . For this case we have

$$\mathcal{Z}_{n \neq 0} = \int \frac{\mathcal{D}\nu \mathcal{D}\nu^*}{(1 + |w_t^{(n)} + \nu|^2)^2} \frac{d^2a}{A} \frac{d\Omega}{4\pi} \frac{d\lambda}{1/\Lambda} e^{-S[w_t^{(n)} + \nu]}. \quad (75)$$

For convenience computing averages, we will normalize the partition somewhat differently, so that we have

$$\mathcal{Z} = \sum_{n=0}^{\infty} \int \frac{\mathcal{D}\nu \mathcal{D}\nu^* d\lambda d^2a d\varphi d\theta \sin \theta}{(1 + |w_t^{(n)} + \nu|^2)^2} \frac{e^{-S(w_t^{(n)} + \nu)}}{\mathcal{Z}_0} \quad (76)$$

The weight in the partition function is the appropriate nonlinear sigma model Euclidean action

$$S(w_t^{(n)} + \nu) = \frac{2\Lambda}{g_\Lambda} \int d^3x \frac{\partial_\mu (w_t^{(n)*} + \nu^*) \partial_\mu (w_t^{(n)} + \nu)}{(1 + |w_t^{(n)} + \nu|^2)^2}. \quad (77)$$

\mathcal{Z}_0 is an arbitrary normalization factor since multiplying the partition function by a constant does not change the physics. We will choose \mathcal{Z}_0 conveniently to make some of our intermediate calculations look simpler.

Of course, the partition function (76) cannot be calculated exactly and we will approximate it by doing perturbation theory about the noninteracting action

$$S_0 = \frac{2\Lambda}{g_\Lambda} \int d\tau d^2x \left[\frac{\partial_\mu w_t^{(n)*} \partial_\mu w_t^{(n)}}{(1 + |w_t^{(n)}|^2)^2} + \partial_\mu \nu^* \partial_\mu \nu \right] \quad (78)$$

$$= S_t + S_\nu$$

where the first action is the unperturbed topolon action and the second is a free spin wave action neglecting the nonlinearity which accounts for the interactions. Then the partition function is

$$\mathcal{Z} = \frac{1}{\mathcal{Z}_0} \sum_{n=0}^{\infty} \int \frac{\mathcal{D}\nu \mathcal{D}\nu^* d\lambda d^2a d\varphi d\theta \sin \theta}{(1 + |w_t^{(n)} + \nu|^2)^2} e^{-S_0 - S_I} \quad (79)$$

$$S_I = S(w_t^{(n)} + \nu) - S_0.$$

We choose

$$\mathcal{Z}_0 = \sum_{n=0}^{\infty} \int \frac{\mathcal{D}\nu \mathcal{D}\nu^* d\lambda d^2a d\varphi d\theta \sin \theta}{(1 + |w_t^{(n)}|^2)^2} e^{-S_0}. \quad (80)$$

We perform perturbation theory in the interaction term by integrating out short distance degrees of freedom. This is done by integrating large momentum and frequency spin waves, those with values between the microscopic cutoff Λ and a lower renormalization scale μ . In order for perturbation theory to be controlled we keep μ close to Λ and improve this perturbation theory by using the renormalization group^{11,12,15,23}. To integrate the short distance topolon configurations we integrate those

with sizes between the lattice constant $1/\Lambda$ and the renormalization distance $1/\mu$. Since μ is close to Λ we expand the action to lowest order in the interaction term and a simple manipulation leads to a partition function with a lower cutoff μ and with effective action correct to first order given by

$$S_{\text{eff}}^\mu = S_0 + \langle S_I \rangle. \quad (81)$$

The average $\langle S_I \rangle$ can be calculated by expanding the interacting action to fourth order in ν and w_t as higher order terms in the expansion contribute higher orders in $\mu - \Lambda$ and are suppressed for μ very close to Λ . This expansion yields the 13 terms

$$\begin{aligned} I &= \frac{2\Lambda}{g_\Lambda} \int d^3x \langle \partial_\mu w_t^n \partial_\mu w_t^{n*} \rangle \\ II &= -\frac{4\Lambda}{g_\Lambda} \int d^3x \langle \partial_\mu w_t^n \partial_\mu w_t^{n*} |w_t^n|^2 \rangle \\ III &= -\frac{4\Lambda}{g_\Lambda} \int d^3x \langle \partial_\mu w_t^n \partial_\mu w_t^{n*} |\nu|^2 \rangle \\ IV &= -\frac{4\Lambda}{g_\Lambda} \int d^3x \langle \partial_\mu w_t^n \partial_\mu w_t^{n*} w_t^n \nu^* \rangle + C.C. \\ V &= \frac{2\Lambda}{g_\Lambda} d^3x \langle \partial_\mu \nu \partial_\mu w_t^{n*} \rangle + C.C. \\ VI &= -\frac{4\Lambda}{g_\Lambda} \int d^3x \langle \partial_\mu \nu \partial_\mu w_t^{n*} |w_t^n|^2 \rangle + C.C. \\ VII &= -\frac{4\Lambda}{g_\Lambda} \int d^3x \langle \partial_\mu \nu \partial_\mu w_t^{n*} |\nu|^2 \rangle + C.C. \\ VIII &= -\frac{4\Lambda}{g_\Lambda} \int d^3x \langle \partial_\mu \nu \partial_\mu w_t^{n*} w_t^n \nu^* \rangle + C.C. \\ IX &= -\frac{4\Lambda}{g_\Lambda} \int d^3x \langle \partial_\mu \nu \partial_\mu w_t^{n*} w_t^{n*} \nu \rangle + C.C. \\ X &= \frac{2\Lambda}{g_\Lambda} d^3x \langle \partial_\mu \nu \partial_\mu \nu^* \rangle \\ XI &= -\frac{4\Lambda}{g_\Lambda} \int d^3x \langle \partial_\mu \nu \partial_\mu \nu^* |w_t^{n*}|^2 \rangle \\ XII &= -\frac{4\Lambda}{g_\Lambda} \int d^3x \langle \partial_\mu \nu \partial_\mu \nu^* |\nu|^2 \rangle \\ XIII &= -\frac{4\Lambda}{g_\Lambda} \int d^3x \langle \partial_\mu \nu \partial_\mu \nu^* w_t^n \nu^* \rangle + C.C. \end{aligned} \quad (82)$$

When we perform the averages, all terms with space derivatives of w_t are irrelevant as the derivatives can be interchanged by a derivative over the position coordinates of the topolons and when averaged, will be smaller than the other terms by a factor of the lattice constant divided by the linear dimensions of the system, a ratio that goes to zero in the thermodynamic limit. Hence we are left only with the terms X to $XIII$. Term $XIII$ being odd in the spin waves averages to zero. The average of term X over short distance gives a constant and is thus irrelevant. We are left with the averaging terms XI and XII . If we go to higher powers in the expansion, there will be similar terms to XI with higher powers of $|w_t^{(n)}|^2$. These can all

be summed. We then obtain that the effective action after averaging over short distance fluctuations is

$$\begin{aligned} & -\frac{4\Lambda}{g_\Lambda} \int d^3x \partial_\mu \nu \partial_\mu \nu^* \langle |\nu|^2 \rangle \\ & + \frac{2\Lambda}{g_\Lambda} \int d^3x \partial_\mu \nu \partial_\mu \nu^* \left\langle \frac{1}{(1 + |w_t^{(n)}|^2)^2} \right\rangle. \end{aligned} \quad (83)$$

The first term, when averaged over short distance fluctuations gives the already calculated Goldstone renormalization

$$\frac{-4\Lambda}{g_\Lambda} \frac{g_\Lambda}{4\pi^2} \left[1 - \frac{\mu}{\Lambda} \right] \int d^3x \partial_\mu \nu \partial_\mu \nu^*. \quad (84)$$

To average over short distance topolons, we first perform the angular average and the average over positions, which gives

$$\left\langle \frac{1}{(1 + |w_t^{(n)}|^2)^2} \right\rangle_{\text{angles+positions}} = \frac{1}{3}. \quad (85)$$

For this average we do not include the weighting factor,

as this factor is independent of the angles and positions. We now integrate over topolon sizes λ from $1/\Lambda$ to $1/\mu$, including the weighting factor to finally obtain

$$\frac{2\Lambda}{3g_\Lambda} \left[1 - \frac{\mu}{\Lambda} \right] \exp \left[-n \frac{8\pi}{g_\Lambda} \right] \int d^3x \partial_\mu \nu \partial_\mu \nu^*. \quad (86)$$

Since there are topolon contributions from $n = 1$ to $n = \infty$, we sum these contributions from 1 to ∞ to obtain

$$\begin{aligned} & \frac{2\Lambda}{3g_\Lambda} \left[1 - \frac{\mu}{\Lambda} \right] \sum_{n=1}^{\infty} \exp \left[-n \frac{8\pi}{g_\Lambda} \right] \int d^3x \partial_\mu \nu \partial_\mu \nu^* \\ & = \frac{2\Lambda}{3g_\Lambda} \left[1 - \frac{\mu}{\Lambda} \right] \frac{1}{e^{8\pi/g_\Lambda} - 1} \int d^3x \partial_\mu \nu \partial_\mu \nu^*. \end{aligned} \quad (87)$$

Both Goldstone and topolon renormalizations are proportional to $1 - \mu/\Lambda$, which can be made as small as possible to make sure the perturbation expansion is controlled. The Goldstone renormalization is linear in the coupling constant, but the topolon contribution is not analytic in the coupling constant around $g_\Lambda = 0$. It is thus non-perturbative. The renormalized action, renormalized spin stiffness and beta function are

$$\begin{aligned} S_{\text{ren}} &= \frac{2\mu}{g_\mu} \int d^3x \frac{\partial_\mu \nu \partial_\mu \nu^*}{(1 + |\nu|^2)^2} \simeq \frac{2\Lambda}{g_\Lambda} \int d^3x \partial_\mu \nu \partial_\mu \nu^* \left\{ 1 - \left(1 - \frac{\mu}{\Lambda} \right) \frac{g_\Lambda}{2\pi^2} + \left(1 - \frac{\mu}{\Lambda} \right) \frac{1}{3(e^{8\pi/g_\Lambda} - 1)} \right\} \\ \rho_s(\mu) &= \frac{\mu}{g_\mu} = \frac{\Lambda}{g_\Lambda} \left\{ 1 - \left(1 - \frac{\mu}{\Lambda} \right) \frac{g_\Lambda}{2\pi^2} + \left(1 - \frac{\mu}{\Lambda} \right) \frac{1}{3(e^{8\pi/g_\Lambda} - 1)} \right\} \\ \beta(g) &= \mu \frac{\partial g}{\partial \mu} \Big|_{\Lambda=\mu} = g - \frac{g^2}{2\pi^2} + \frac{g}{3(e^{8\pi/g} - 1)}. \end{aligned} \quad (88)$$

The last term in the spin stiffness and in the beta function is the contribution from the topolons and the rest is the contribution of the spin waves to fourth order in the coupling constant. The coupling constant at the quantum critical point $g_c \simeq 23.0764$ is obtained from $\beta(g_c) = 0$.

The important point is that the topolons add a new term to the beta function calculated above. This will lead to modification of the critical properties beyond those calculated from the spin wave expansion only. Below we estimate exponents from our approxiamtions. These exponents will not be suffiently accurate for two reasons. First, we calculated our spin wave expansion to first order in the coupling constant g . This will lead to large inaccuracies as g is not small. On the other hand, this would be easy to improve by including higher order corrections obtainable from the $d = 4 - \epsilon$ expansion, which is Borel summable and thus very accurate as far as the spin wave effects concern²³. Inclusion of these terms will not change the new term we found from the topolons and hence will not affect the fact that there is new critical physics coming from superpositions of skyrmions and

antiskyrmions. On the other hand such corrections are necessary for accurate enough exponents.

Another issue is that there might be more exact solutions that correspond to superpositions of equal numbers of skyrmions and antiskyrmions. We found no other solutions, but it is expected that there should be solutions that are time dependent in which the skyrmions and antiskyrmions are in motion. These need to be included in order to have the most accurate possible exponents, but they do not invalidate the physics we have found: that skyrmions and antiskyrmions modify critical properties. They further add to the physics we have found in the present work. Finally, for completeness we compare below with other approaches that people use to calcualte exponents. Such approaches are the $1/N$ expansion and the $d = 2 + \epsilon$ expansion. These are not as accurate as the $d = 4 - \epsilon$ expansion, which seems to give the accepted Heisenberg exponents²³, but this last expansion cannot capture topological contributions as these exist only in two spatial dimensions.

The correlation length at scale μ is given by¹⁰

$$\xi \sim \frac{1}{\mu} \exp \left[\int_{g_c}^{g_\mu} \frac{dg}{\beta(g)} \right] \sim (g_c - g_\mu)^{1/\beta'(g_c)}. \quad (89)$$

The correlation length exponent is $\nu = -1/\beta'(g_c) \equiv -(d\beta/dg|_{g=g_c})^{-1}$. Including topolon contributions, it evaluates to $\nu = 0.9297$. The $d = 2 + \epsilon$ expansion of the $O(N)$ vector model, which agrees with the $1/N$ expansion for large N , gives $\nu = 0.5$ ²³. We note that our value is larger than the accepted numerical evaluations of critical exponents in the Heisenberg model, $\nu = 0.71125$ ²⁴, but about as close to this accepted Heisenberg value than the $2 + \epsilon$ expansion or the $1/N$ expansion. We conjecture that the difference between our value and the Heisenberg value is real and attributable to quantum critical degrees of freedom.

Goldstone renormalizations of the ordering direction $\sigma = n_3$, and hence of the anomalous dimension η , are notoriously inaccurate. The one loop approximation leads to a value of $\eta = 2$, thousands of percent different from the accepted numerical Heisenberg value of $\eta \simeq 0.0375$ ²⁴. The large N approximation, which sums bubble diagrams, is a lot more accurate. To order $1/N$ one obtains $\eta = 8/(3\pi^2 N) \simeq 0.09$ for $N = 3$. We now calculate the value of η from topological nontrivial configurations

$$\begin{aligned} \langle n_3^2 \rangle &= Z = 1 - \langle n_1^2 + n_2^2 \rangle \\ &= 1 - \left\langle \frac{4|w|^2}{(1+|w|^2)^2} \right\rangle \simeq 1 - \frac{2}{3} \frac{1-\mu/\Lambda}{e^{8\pi/g_\Lambda} - 1} \\ \Rightarrow \eta(g) &= \frac{\mu}{Z} \frac{\partial Z}{\partial \mu} \Big|_{\Lambda=\mu} \simeq \frac{2}{3(e^{8\pi/g} - 1)}. \end{aligned} \quad (90)$$

For the anomalous dimension at the quantum critical point we obtain $\eta(g_c) = 0.3381$. On the other hand, we have seen that spin wave contributions tend to give quite large and nonsensical values of η . In fact so large as to wash out the momentum dependence of the propagator. Hence, to calculate η , spin wave contributions prove to be tough to control. Our calculation gives a value quite larger than the accepted numerical value. We have recently calculated³⁰ the unique value of η that follows from quantum critical fractionalization into spinons and find $\eta = 1$. While our value obtained from topolons is far from 1, it is a lot closer than the accepted numerical Heisenberg value and the $1/N$ value.

The main purpose of this work is to look for possible new physics due to intrinsic critical excitations of quantum critical points. In this article we studied the approach to the quantum critical point from the Néel ordered phase. We have uncovered some things that were not known before. One thing which has been hinted at before was that skyrmion excitations are relevant to the quantum critical point. We have provided strong evidence that skyrmion fluctuations contribute to the quantum critical point. For the first time, we found new exact solutions that corresponds to superpositions of equal number of skyrmions and antiskyrmions. Also for the

first time, we developed and calculated the inclusion of these topological effects in the renormalization group to yield critical exponents.

APPENDIX A: STEREOGRAPHY AND TOPOLOGY

1. 2 + 1 D Antiferromagnets

In the present work we study 2 dimensional $O(3)$ quantum antiferromagnets on bipartite lattices as described by the the Heisenberg Hamiltonian

$$H = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j \quad (A1)$$

with $J > 0$ and $\langle ij \rangle$ meaning that ij are next neighbors. Haldane²⁵ showed that in the large S limit, the low energy universal physics of the Heisenberg antiferromagnet is equivalent to that given by the $O(3)$ nonlinear sigma model described by the Lagrangian and action

$$L = \frac{\Lambda}{2g} \int d^2x \eta^{\mu\nu} \partial_\mu \vec{n} \cdot \partial_\nu \vec{n} = \frac{\Lambda}{2g} \int d^2x \partial^\mu \vec{n} \cdot \partial_\mu \vec{n} \quad (A2)$$

$$S = \int dt L, \quad (A3)$$

where $\eta^{\mu\nu}$ is the $2 + 1$ Lorentz metric and \vec{n} is a 3 dimensional unit vector, $\vec{n} \cdot \vec{n} = 1$, that represents the sublattice magnetization. Physically the inverse coupling constant is proportional to the “spin” or magnetization stiffness ρ_s . The large S identification of the Heisenberg and nonlinear sigma model holds because the amplitude fluctuations of the spins are irrelevant for large S ²⁵. Hence as long as the amplitude fluctuations are irrelevant to the long distance physics, the $O(3)$ nonlinear sigma model will be an apt description of antiferromagnets *regardless of S*. One expects amplitude fluctuations to be less relevant for lower dimensionality.

A very useful way of describing the $O(3)$ nonlinear sigma model is through the stereographic projection⁸:

$$n^1 + in^2 = \frac{2w}{|w|^2 + 1}, \quad n^3 = \frac{1 - |w|^2}{1 + |w|^2}, \quad w = \frac{n^1 + in^2}{1 + n^3}. \quad (A4)$$

The stereographic projection maps the sphere subtended by the Néel field or staggered magnetization \vec{n} to the complex w plane by placing the Néel sphere below the w plane, with the north pole touching the center of the plane. If one draws a straight line joining the south pole and the point on the sphere corresponding to the Néel field, then the mapping to the w plane of this Néel direction is given by extending the mentioned line until it intersects the w plane.

In terms of w the Lagrangian is

$$\begin{aligned} L &= \frac{2\Lambda}{g} \int d^2x \frac{\partial^\mu w \partial_\mu w^*}{(1+|w|^2)^2} \\ &= \frac{2\Lambda}{g} \int d^2x \frac{\partial_0 w \partial_0 w^* - 2\partial_z w \partial_{z^*} w^* - 2\partial_{z^*} w \partial_z w^*}{(1+|w|^2)^2}, \end{aligned} \quad (\text{A5})$$

where $z = x + iy$ and $z^* = x - iy$ is its conjugate. The classical equations of motion which follow by stationarity of the classical action are $\square \vec{n} = 0$, which in terms of the stereographic variable w are

$$\begin{aligned} \square w &= \frac{2w^*}{1+|w|^2} \partial^\mu w \partial_\mu w \text{ or} \\ \partial_0^2 w - 4\partial_z \partial_{z^*} w &= \frac{2w^*}{1+|w|^2} [(\partial_0 w)^2 - 4\partial_z w \partial_{z^*} w]. \end{aligned} \quad (\text{A6})$$

The quantum mechanics of the $O(3)$ nonlinear sigma model is achieved either via path integral or canonical quantization. The last is performed by defining the momentum conjugate to \vec{n} , or to w and w^* , by

$$\begin{aligned} \vec{\Pi}(t, \vec{x}) &\equiv \frac{\delta L}{\delta \partial_0 \vec{n}(t, \vec{x})} \\ \Pi^*(t, \vec{x}) &\equiv \frac{\delta L}{\delta \partial_0 w(t, \vec{x})}, \quad \Pi(t, \vec{x}) \equiv \frac{\delta L}{\delta \partial_0 w^*(t, \vec{x})}. \end{aligned} \quad (\text{A7})$$

and then imposing canonical commutation relations among the momenta and coordinates. Due to the nonlinear constraint $\vec{n} \cdot \vec{n} = 1$, the momentum $\vec{\Pi}$ is an angular momentum satisfying the $SU(2)$ algebra:

$$\vec{\Pi} \cdot \vec{n} = 0, \quad \vec{\Pi} \times \vec{\Pi} = i\vec{\Pi}. \quad (\text{A8})$$

The Hamiltonian is then given by

$$\begin{aligned} H &= \int d^2x \left(\vec{\Pi} \cdot \partial_0 \vec{n} - L \right) \\ &= \int d^2x \left[\frac{g}{2\Lambda} \vec{\Pi}^2 + \frac{\Lambda}{2g} \partial_i \vec{n} \cdot \partial_i \vec{n} \right] \\ &= \int d^2x (\Pi^* \cdot \partial_0 w + \Pi \cdot \partial_0 w^* - L) \\ &= \int d^2x \left[\frac{g}{2\Lambda} (1+|w|^2)^2 \Pi^* \Pi + \frac{2\Lambda}{g} \frac{\partial_i w \partial_i w^*}{(1+|w|^2)^2} \right] \\ &= \int d^2x \left[\frac{g}{2\Lambda} (1+|w|^2)^2 \Pi^* \Pi \right. \\ &\quad \left. + \frac{4\Lambda}{g} \frac{(\partial_z w \partial_{z^*} w^* \partial_{z^*} w \partial_z w^*)}{(1+|w|^2)^2} \right]. \end{aligned} \quad (\text{A9})$$

The Heisenberg equations of motion that follow from this Hamiltonian, when properly ordered, are identical to the classical equations. There are ordering ambiguities in this Hamiltonian. The usual prescription to deal with the ambiguities is by symmetrization, but the correct order can only be determined by comparison with experiment if there is a measurement that is sensitive to the operator order. Most results are insensitive to these ordering ambiguities as they only introduce short distance modifications to the physics.

2. Excitations of the Néel Ordered Phase of $O(3)$ Nonlinear Sigma Model

We remind the reader that classically the lowest energy state is Néel ordered for all $g < \infty$, i.e. the spin stiffness, ρ_s , is never zero. Quantum mechanically the situation is different. In 2 + 1 and higher dimensions, quantum mechanical fluctuations cannot destroy the Néel order for the bare coupling constant less than some critical value g_c ^{10,11,12}. At g_c the renormalized long-distance, low-energy coupling constant diverges¹⁰, i.e. the system loses all spin stiffness. At such a point quantum fluctuations destroy the Néel order in the ground state as the renormalized stiffness vanishes. In the present section we concentrate in the excitations of the Néel ordered phase.

3. Magnons

Linearization of the equations of motion leads to the low energy excitations of the sigma model (magnons in the Néel phase and triplons in the disordered phase) when quantized. We now turn our attention to the Néel ordered phase. When the system Néel orders, \vec{n} , or equivalently w , will acquire an expectation value:

$$\langle n^a \rangle = -\delta^{3a}, \quad \left\langle \frac{1}{w} \right\rangle = 0. \quad (\text{A10})$$

where we have chosen the order parameter in the -3 -direction as it will always point in an arbitrary, but fixed direction. Small fluctuations about the order parameter

$$\frac{1}{w} = \nu \quad (\text{A11})$$

are the magnons or Goldstone excitations of the Néel phase. To leading order the magnon Lagrangian is

$$\begin{aligned} L &= \frac{2\Lambda}{g} \int d^2x \frac{\partial^\mu \nu \partial_\mu \nu^*}{(1+|\nu|^2)^2} \simeq \frac{2\Lambda}{g} \int d^2x \partial^\mu \nu \partial_\mu \nu^* \\ &= \frac{2\Lambda}{g} \int d^2x (\partial_0 \nu \partial_0 \nu^* - 2\partial_z \nu \partial_{z^*} \nu^* - 2\partial_{z^*} \nu \partial_z \nu^*) \end{aligned} \quad (\text{A12})$$

leading to the equations of motion

$$\square \nu = 0, \quad \partial_0^2 \nu - 4\partial_z \partial_{z^*} \nu = 0. \quad (\text{A13})$$

The linearized excitations of the Néel phase have relativistic dispersion that vanishes at long wavelengths as dictated by Goldstone's theorem²⁶. The magnons are of course spin 1 particles. They have only 2 polarizations as they are transverse to the Néel order.

4. Skyrmions

The Goldstones are not the only excitations of the ordered phase in the nonlinear sigma model. Since the

1970's, it has been known that exact or approximate time independent solutions of the classical equations of motion, when stable against quantum fluctuations, are quantum particle excitations of the system¹⁹. The non-linear sigma model possesses time independent solutions of a topological nature^{8,20}. These excitations are disordered at finite length scales but relax into the Néel state far away:

$$\lim_{|\vec{x}| \rightarrow \infty} \vec{n} = (0, 0, -1), \quad \lim_{|\vec{x}| \rightarrow \infty} w = \infty. \quad (\text{A14})$$

They consist in the order parameter rotating a number of times as one moves from infinity toward a fixed but arbitrary position in the plane. Since two dimensional space can be thought of as an infinite 2 dimensional sphere, the excitations fall in homotopy classes of a 2D sphere into a 2D sphere: $S^2 \rightarrow S^2$. The topological excitations are thus defined by the number of times they map the 2D sphere into itself. They are thus characterized by the Jacobian

$$q = \frac{1}{8\pi} \int d^2x \epsilon^{ij} \vec{n} \cdot \partial_i \vec{n} \times \partial_j \vec{n}. \quad (\text{A15})$$

or

$$\begin{aligned} q &= \frac{i}{2\pi} \int d^2x \frac{\epsilon^{ij} \partial_i w \partial_j w^*}{(1 + |w|^2)^2} \\ &= \frac{1}{\pi} \int d^2x \frac{\partial_z w \partial_{z^*} w^* - \partial_{z^*} w \partial_z w^*}{(1 + |w|^2)^2}. \end{aligned} \quad (\text{A16})$$

The number q will be an integer measuring how many times the n -sphere gets mapped into the infinite 2D sphere corresponding to the plane where the spins live. If we define the space-time current

$$J^\mu = \frac{1}{8\pi} \epsilon^{\mu\nu\sigma} \vec{n} \cdot \partial_\nu \vec{n} \times \partial_\sigma \vec{n} = \frac{i}{2\pi} \epsilon^{\mu\nu\sigma} \frac{\partial_\nu w \partial_\sigma w^*}{(1 + |w|^2)^2}, \quad (\text{A17})$$

it is easily seen that it is conserved $\partial_\mu J^\mu = 0$ and that the charge associated with it is our topological charge:

$$q = \int d^2x J^0. \quad (\text{A18})$$

Thus q is a conserved quantum number. These topological field configurations were originally discovered by Skyrme²¹ and are called skyrmions. The conserved charge is thus the skyrmion number.

From the expressions for the charge q and for the Hamiltonian, it is easily seen^{8,20} that $E \geq 4\pi\Lambda|q|/g$. We see that we can construct skyrmions with $q > 0$ by imposing the condition

$$\partial_{z^*} w = 0, \quad (\text{A19})$$

that is w is a function of z only. Since the magnetization, \vec{n} or w , is a continuous function of z , the worst singularities it can have are poles. The skyrmions will have a location given by the positions of the poles or

of the zeros of w . Far away from its position, the field configuration will relax back to the original Néel order. Therefore we have the boundary condition $w(\infty) = \infty$, which implies

$$w = \frac{1}{\lambda^q} \prod_{i=1}^q (z - a_i) \quad (\text{A20})$$

This can easily be checked to have charge q and energy $4\pi\Lambda q/g$. λ^q is the arbitrary size and phase of the configuration and a_i are the positions of the skyrmions that constitute the multiskyrmion configuration. The energy is independent of the size and phase due to the conformal invariance of the configuration. We remark that since the multiskyrmions energy is the sum of individual skyrmion energies, the skyrmions do not interact among themselves⁸. An example of the explicit calculation of the charge and energy for a disk skyrmion is shown in subsection A 6. Similarly, the multiantiskyrmion configuration can be shown to be

$$w = \frac{1}{(\lambda^*)^q} \prod_{i=1}^q (z^* - a_i^*) \quad (\text{A21})$$

with charge $-q$ and energy $4\pi\Lambda q/g$.

We have just studied the skyrmion and antiskyrmion configurations which relax to a Néel ordered configuration in the -3 direction far away from their positions. We shall call them -3 -skyrmions. The skyrmion direction is given by the boundary conditions as $z \rightarrow \infty$. For example, $(z-a)/\lambda$ gives $n^a(\infty) = -\delta^{3a}$, so it is a -3 -skyrmion. The $+3$ -skyrmion is $\lambda/(z-a)$. The $+1$ -skyrmion is $(z-a)/(z-b)$. The -1 -skyrmion is $-(z-a)/(z-b)$. The $+2$ -skyrmion is $i(z-a)/(z-b)$. The -2 -skyrmion is $-i(z-a)/(z-b)$. Because of the rotational invariance of the underlying theory, they are all kinematically equivalent. They are not dynamically equivalent since a Néel ordered ground state has skyrmions and antiskyrmions corresponding to its ordering direction as excitations.

That the skyrmion configurations behave like particles follows easily by making them time dependent and examining their dynamics. We do so for a single skyrmion here:

$$w = \frac{z - a}{\lambda}. \quad (\text{A22})$$

We make the skyrmion time dependent by allowing it to move (making its position, $a(t)$, time dependent), and become fatter or slimmer with time (making its size, $\lambda(t)$, time dependent). We substitute this time dependent configuration in the Lagrangian and obtain in subsection A 7:

$$L = \frac{2\pi\Lambda}{g} |\dot{a}|^2 - \frac{4\pi\Lambda}{g}. \quad (\text{A23})$$

Since the skyrmion Lagrangian acquired a term proportional to the skyrmion velocity squared, a kinetic energy term, we see that the skyrmion behaves like a free particle of mass $4\pi\Lambda/g$ with an excitation gap of $4\pi\Lambda/g$. The

skyrmion position is a dynamical variable. On the other hand, the conformal parameter λ does not have dynamics as it has infinite mass in the thermodynamic limit, see subsection A 7. The conformal parameter is thus an arbitrary constant making the skyrmion configuration conformally invariant even when we allow time dependence of the configuration. Even though the sigma model does not have a microscopic length, real antiferromagnets will have a microscopic length as a consequence of amplitude fluctuations. We thus physically expect $|\lambda|$ to be cutoff at small values by a coherence length ξ . The long distance physics is, of course, insensitive to this cutoff.

5. Skyrmion-Antiskyrmion States

Since the charge or skyrmion number is conserved, a configuration with nonzero skyrmion number cannot be excited out of the ground state in the absence of an external probe that couples to skyrmion number. Therefore, skyrmions and antiskyrmions will be created in equal numbers. We thus have to study the interaction between skyrmions and antiskyrmions. A skyrmion-antiskyrmion configuration, which is not a solution to the equations of motion, is given by

$$w = \frac{1}{\lambda^2}(z - a)(z^* - b^*) . \quad (\text{A24})$$

The energy of this static configuration is given *exactly* by

$$\begin{aligned} E_s &= \frac{4\pi\Lambda}{g} \\ &+ \frac{\Lambda |a - b|^4}{g |\lambda|^4} \int_0^\infty \frac{r K\left(\frac{4r}{(r+1)^2}\right)}{(1 + \frac{r^2|a-b|^4}{16|\lambda|^4})^2(r+1)} dr . \end{aligned} \quad (\text{A25})$$

where $K(x)$ is the Jacobian elliptic function²⁷. We reproduce the details of the calculation in subsection A 8 because it has been calculated or approximated incorrectly in previous works^{8,28,29}. The skyrmion-antiskyrmion interaction, or potential energy, is given by the difference between the static energy E_s and the sum of the energies of the isolated skyrmion and skyrmion, $V = E_s - 8\pi\Lambda/g$.

The skyrmion-antiskyrmion interaction has many interesting features. As the distance between the skyrmion and antiskyrmion becomes small compared to the size of the configuration, $|a - b|/|\lambda| \ll 1$, their interaction is very soft; the energy goes like

$$V \simeq -\frac{4\pi\Lambda}{g} + \frac{\pi^2\Lambda}{2g} \frac{|a - b|^2}{|\lambda|^2} . \quad (\text{A26})$$

At short distances the skyrmions and antiskyrmions are bound by a harmonic potential. The minimum of this classical energy occurs when the skyrmion-antiskyrmion form a bound state with zero “separation” between the skyrmion and antiskyrmion, or equivalently infinite conformal size, i.e. $|a - b|/|\lambda| = 0$. This bound state

resonance has energy $-4\pi\Lambda/g$, or a binding energy of $4\pi\Lambda/g$. Therefore the skyrmion and antiskyrmion gaps get halved. When this skyrmion-antiskyrmion configuration has a large but finite size, i.e. $|\lambda|/|a - b| \gg 1$, the potential between the skyrmion and antiskyrmion is very soft and vanishes when the configuration has arbitrarily large size. In this limit the skyrmion and antiskyrmion do not interact despite being “bound”.

At large distances or small size, $|a - b|/|\lambda| \gg 1$, the interaction is approximately

$$V \simeq \frac{64\pi\Lambda}{g} \frac{|\lambda|^4}{|a - b|^4} \ln\left(\frac{|a - b|}{2|\lambda|}\right) . \quad (\text{A27})$$

At large enough distances the skyrmion and antiskyrmion are almost free and repel each other with an interaction that vanishes at infinitely large separations. We see that the skyrmion-antiskyrmion potential is attractive at short distances or large sizes, while at larger distances or small size it goes to a maximum energy which is higher than 0 and then vanishes at infinity. In order to unbind them one has to at least supply an energy $4\pi\Lambda/g$. Classically one would have to supply enough energy to get over the potential energy hump, but quantum mechanically one can, of course, tunnel through the barrier.

Contrary to pure skyrmion or pure antiskyrmion configurations, the skyrmion-antiskyrmion configurations are not stationary solutions of the equations of motion. Therefore the dynamics will not be restricted to center of mass motion alone. In order to study the dynamics of the skyrmion and antiskyrmion configurations we allow motion of the positions of the skyrmion, $a(t)$, and the antiskyrmion, $b(t)$, and permit time dependence of the conformal parameter, $\lambda(t)$. We substitute this time dependent configuration in the sigma model Lagrangian in subsection A 9 and obtain the kinetic energy part of the Lagrangian to be

$$T = \frac{m_{ab}}{2} (|\dot{a}|^2 + |\dot{b}|^2) + \frac{m_\lambda}{2} |\dot{\lambda}|^2 \quad (\text{A28})$$

with

$$\begin{aligned} m_\lambda &= \frac{\Lambda}{2g} \frac{|a - b|^6}{|\lambda|^6} \\ &\times \int_0^\infty \frac{R^3 K\left(\frac{4R}{(R+1)^2}\right) dR}{(1 + |a - b|^4 R^2 / 16|\lambda|^4)^2(R+1)} \\ m_{ab} &= \frac{4\Lambda}{g} \int_0^\infty \int_0^{2\pi} r^3 dr d\theta \times \\ &\frac{1}{[1 + r^2(r^2 + |a - b|^2 / |\lambda|^2 - 2r|a - b| \cos \theta) / |\lambda|]^2} \end{aligned} \quad (\text{A29})$$

with $K(x)$ the Jacobian elliptic function²⁷. At short distances, or large sizes, $|a - b| \ll |\lambda|$, the a , b and λ masses have the asymptotic behavior

$$m_\lambda \simeq \frac{4\pi^2\Lambda}{g}, \quad m_{ab} \simeq \frac{2\pi\Lambda}{g} . \quad (\text{A30})$$

In this limit the mass of the skyrmion and antiskyrmion is equal to 1/2 the mass of an isolated skyrmion or anti-skyrmion. At large distances, or small sizes, $|a - b| \gg |\lambda|$, the masses go like

$$m_\lambda \simeq \frac{64\pi\Lambda}{g} \frac{|\lambda|^2}{|a - b|^2} \ln \left(\frac{|a - b|^2}{4|\lambda|^2} \right), m_{ab} \simeq \frac{2^9\pi\Lambda}{g} \frac{|\lambda|^8}{|a - b|^8}. \quad (\text{A31})$$

The Lagrangian that describes the dynamics of the skyrmion-antiskyrmion configuration is thus

$$\begin{aligned} L = & \frac{m_{ab}}{4} \left| \frac{d}{dt} (a + b) \right|^2 + \frac{m_{ab}}{4} \left| \frac{d}{dt} (a - b) \right|^2 \\ & + \frac{m_\lambda}{2} |\dot{\lambda}|^2 - V \left(\frac{|a - b|}{|\lambda|} \right). \end{aligned} \quad (\text{A32})$$

We see that the center of mass coordinate decouples as required by the translational invariance of the system. Contrary to the pure skyrmion configurations, here the conformal parameter, λ , has dynamics and is not an arbitrary parameter. That is, the skyrmion-antiskyrmion is not conformally invariant.

6. The Diskyrmiion

In the present subsection we explicitly check that a diskyrmiion configuration has charge 2 and energy equal to the sum of the energies of the two single skyrmions that constitute the diskyrmiion. This is important because there are false claims²⁸ in the literature that multiskyrmiion configurations interact through logarithmic potentials. Above we concluded that since a multi-skyrmion configuration has an energy that is the sum of the skyrmions that constitute it, skyrmions do not interact. This was originally concluded by Gross⁸. In the present section we show this by explicit calculation for the diskyrmiion energy.

The diskyrmiion configuration is

$$w = \frac{1}{\lambda^2} (z - a)(z - b). \quad (\text{A33})$$

The charge is given by

$$q = \frac{4}{\pi} \int d^2x \frac{|z - (\tilde{a} + \tilde{b})/2|^2}{(1 + |z - \tilde{a}|^2|z - \tilde{b}|^2)^2}. \quad (\text{A34})$$

where we have rescaled z and defined $\tilde{a} = a/\lambda$ and $\tilde{b} = b/\lambda$ in order to absorb the arbitrary size λ . The energy is given by

$$E = \frac{16\Lambda}{g} \int d^2x \frac{|z - (\tilde{a} + \tilde{b})/2|^2}{(1 + |z - \tilde{a}|^2|z - \tilde{b}|^2)^2}. \quad (\text{A35})$$

In order to calculate the energy and charge we define

$$A \equiv \frac{\tilde{a} + \tilde{b}}{2}, \quad B \equiv \frac{\tilde{a} - \tilde{b}}{2}, \quad (\text{A36})$$

and make the change of origin $z \rightarrow z + A$, to obtain that the energy and charge are $E = 16\Lambda I/g$ and $q = 4I/\pi$, where I is the integral

$$\begin{aligned} I = & \frac{1}{2} \int \frac{|z|^2 dz dz^*}{[1 + |z + B|^2|z - B|^2]^2} \\ = & \frac{1}{2} \int \frac{z dz z^* dz^*}{\{1 + [z^2 - B^2][(z^*)^2 - (B^*)^2]\}^2} \\ = & \frac{1}{8} \int \frac{d[z^2 - B^2] d[(z^*)^2 - (B^*)^2]}{\{1 + [z^2 - B^2][(z^*)^2 - (B^*)^2]\}^2} \\ = & \frac{1}{4} \int \frac{du du^*}{[1 + |u|^2]^2}. \end{aligned} \quad (\text{A37})$$

To obtain the last equality we made the variable change $u = z^2 - B^2$. The factor of 2 arises because u is linearly related to z^2 , so one must cover the u complex plane twice in order to cover the z complex plane once. Going over to polar coordinates of the u plane we get

$$I = \frac{1}{2} \int \frac{r dr d\theta}{(1 + r^2)^2} = \frac{\pi}{2}. \quad (\text{A38})$$

We finally obtain

$$q = \frac{4}{\pi} I = 2, \quad E = \frac{16\Lambda}{g} I = \frac{8\pi\Lambda}{g} = \frac{4\pi\Lambda}{g} q \quad (\text{A39})$$

as expected.

7. Skyrmion Kinetic Energy

When we substitute the time dependent skyrmion

$$w = \frac{z - a(t)}{\lambda(t)}. \quad (\text{A40})$$

into the sigma model Lagrangian

$$\begin{aligned} L = & \frac{2\Lambda}{g} \int d^2x \frac{1}{(1 + |w|^2)^2} \\ & \times (\partial_0 w \partial_0 w^* - 2\partial_z w \partial_{z^*} w^* - 2\partial_{z^*} w \partial_z w^*) \end{aligned} \quad (\text{A41})$$

we obtain

$$\begin{aligned} L = & -\frac{4\pi\Lambda}{g} + \frac{2\Lambda}{g} \int d^2x \frac{1}{(1 + |w|^2)^2} \\ & \times \left(\frac{|\dot{\lambda}|^2}{|\lambda|^2} |w|^2 + \frac{|\dot{a}|^2}{|\lambda|^2} + \frac{\dot{\lambda} \dot{a}^*}{|\lambda|^2} w + \frac{\dot{\lambda}^* \dot{a}}{|\lambda|^2} w^* \right). \end{aligned} \quad (\text{A42})$$

First we evaluate the integral

$$\int d^2x \frac{w}{(1 + |w|^2)^2} = |\lambda|^2 \int \frac{r^2 e^{i\theta} dr d\theta}{(1 + r^2)^2} = 0 \quad (\text{A43})$$

where we made the variable change $z \rightarrow z + a$, the conformal transformation $z \rightarrow \lambda z$, and went to polar coordinates of the complex z plane. Similarly we have

$$\int d^2x \frac{w^*}{(1 + |w|^2)^2} = |\lambda|^2 \int \frac{r^2 e^{-i\theta} dr d\theta}{(1 + r^2)^2} = 0. \quad (\text{A44})$$

The skyrmion Lagrangian is then

$$\begin{aligned} L &= -\frac{4\pi\Lambda}{g} + \frac{2\Lambda}{g} \int d^2x \\ &\times \left\{ \frac{|\dot{a}|^2}{|\lambda|^2(1+|w|^2)^2} + \frac{|\dot{\lambda}|^2}{|\lambda|^2(1+|w|^2)^2} \frac{|w|^2}{(1+|w|^2)^2} \right\} \\ &= \frac{m_a}{2} |\dot{a}|^2 + \frac{m_\lambda}{2} |\dot{\lambda}|^2 - \frac{4\pi\Lambda}{g}. \end{aligned} \quad (\text{A45})$$

We see that there is a kinetic energy term for a with mass

$$\begin{aligned} m_a &= \frac{4\Lambda}{g|\lambda|^2} \int d^2x \frac{1}{(1+|w|^2)^2} \\ &= \frac{4\Lambda}{g} \int \frac{rdrd\theta}{(1+r^2)^2} = \frac{4\pi\Lambda}{g} \end{aligned} \quad (\text{A46})$$

and a kinetic energy term for λ with mass

$$\begin{aligned} m_\lambda &= \frac{4\Lambda}{g|\lambda|^2} \int d^2x \frac{|w|^2}{(1+|w|^2)^2} \\ &= \frac{4\Lambda}{g} \int \frac{r^3drd\theta}{(1+r^2)^2} \\ &= \frac{4\Lambda}{g} \left[\pi \ln(1+r^2) \Big|_0^\infty - \pi \right] = \infty. \end{aligned} \quad (\text{A47})$$

Since the mass for λ is infinite, $\dot{\lambda} = 0$ exactly in order not to pay an infinite kinetic energy cost. Thus λ is not a dynamical variable, but a constant arbitrary parameter. This is true classically and quantum mechanically. Quantum mechanically, the term

$$\frac{|p_\lambda|^2}{2m_\lambda} \rightarrow 0 \quad \text{as} \quad m_\lambda \rightarrow \infty \quad (\text{A48})$$

with p_λ the momentum conjugate to λ . Since the p_λ is the only quantity that does not commute with λ in the Hamiltonian H , it immediately follows that

$$\dot{\lambda} = -\frac{i}{\hbar} [\lambda, H] = 0 \quad (\text{A49})$$

in the infinite mass limit.

8. Skyrmion-Antiskyrmion Static Energy

We now calculate the skyrmion number q and energy E of the skyrmion-antiskyrmion static configuration

$$w = \frac{1}{\lambda^2} (z - a)(z^* - b^*). \quad (\text{A50})$$

The charge is given by

$$q = \frac{1}{\pi} \int d^2x \frac{|z - \tilde{b}|^2 - |z - \tilde{a}|^2}{(1 + |z - \tilde{a}|^2|z - \tilde{b}|^2)^2} \quad (\text{A51})$$

where we have made the conformal transformation $z \rightarrow \lambda z$ and defined $\tilde{a} = a/\lambda$ and $\tilde{b} = b/\lambda$ in order to absorb

the arbitrary conformal parameter λ . The energy is given by

$$E = \frac{4\Lambda}{g} \int d^2x \frac{|z - \tilde{b}|^2 + |z - \tilde{a}|^2}{(1 + |z - \tilde{a}|^2|z - \tilde{b}|^2)^2}. \quad (\text{A52})$$

As in subsection A 6, in order to calculate the energy and charge we define

$$A \equiv \frac{\tilde{a} + \tilde{b}}{2}, \quad B \equiv \frac{\tilde{a} - \tilde{b}}{2} \quad (\text{A53})$$

and make the change of origin $z \rightarrow z + A$, to obtain that the energy and charge are

$$\begin{aligned} E &= \frac{8\Lambda}{g} \int \frac{1}{2} \frac{|z|^2 dz dz^*}{(1 + |z + B|^2|z - B|^2)^2} \\ &+ \frac{4\Lambda}{g} |B|^2 \int \frac{dz dz^*}{(1 + |z + B|^2|z - B|^2)^2} \\ q &= \frac{1}{2\pi} \int dz dz^* \frac{(|z + B|^2 - |z - B|^2)}{(1 + |z + B|^2|z - B|^2)^2}. \end{aligned} \quad (\text{A54})$$

The charge is easily seen to be zero. It is the subtraction of two integrals. If on the second integral we take $z \rightarrow -z$, it becomes equal to the first and thus cancels it upon subtraction. Therefore $q = 0$ for the skyrmion-antiskyrmion configuration as expected. The first term in the energy was evaluated in subsection A 6. We thus have

$$E = \frac{4\pi\Lambda}{g} + \frac{4\Lambda}{g} |B|^2 \int \frac{dz dz^*}{[1 + |z + B|^2|z - B|^2]^2}. \quad (\text{A55})$$

Making the variable change $u = z^2 - B^2$, we get

$$\begin{aligned} E &= \frac{4\pi\Lambda}{g} + \frac{2\Lambda}{g} |B|^2 \int \frac{dudu^*}{|u + B^2| [1 + |u|^2]^2} \\ &= \frac{4\pi\Lambda}{g} + \frac{4\Lambda}{g} |B|^2 \int r dr d\theta \\ &\times \frac{1}{(1+r^2)^2 \sqrt{r^2 + |B|^4 + 2r|B|^2 \cos \theta}} \end{aligned} \quad (\text{A56})$$

where for the last equality we went to polar coordinates of the complex u plane. Now

$$\begin{aligned} &\int_0^{2\pi} \frac{d\theta}{\sqrt{r^2 + |B|^4 + 2r|B|^2 \cos \theta}} \\ &= 2 \int_0^\pi \frac{d\theta}{\sqrt{r^2 + |B|^4 + 2r|B|^2 \cos \theta}} \\ &= 2 \int_0^\pi \frac{d\theta}{\sqrt{r^2 + |B|^4 + 2r|B|^2 - 4r|B|^2 \sin^2(\theta/2)}} \\ &= 4 \int_0^{\pi/2} \frac{d\phi}{\sqrt{r^2 + |B|^4 + 2r|B|^2 - 4r|B|^2 \sin^2 \phi}} \\ &= \frac{4}{r + |B|^2} K \left(\frac{4r|B|^2}{(r + |B|^2)^2} \right) \end{aligned} \quad (\text{A57})$$

where $K(x)$ is the Jacobian elliptic function²⁷. After making the variable change $R = |B|^2 r$, we then have

$$\begin{aligned} E &= \frac{4\pi\Lambda}{g} + \frac{16\Lambda}{g} |B|^4 \int_0^\infty \frac{K\left(\frac{4R}{(R+1)^2}\right) R dR}{(R+1)(1+|B|^4 R^2)^2} \\ &= \frac{4\pi\Lambda}{g} + \frac{\Lambda}{g} \frac{|a-b|^4}{|\lambda|^4} \int_0^\infty K\left(\frac{4R}{(R+1)^2}\right) R dR \quad (\text{A58}) \\ &\times \frac{1}{[R+1][1+|a-b|^4 R^2/(16|\lambda|^4)]^2}. \end{aligned}$$

Asymptotic approximations yield to leading order

$$\begin{aligned} E &\simeq \frac{4\pi\Lambda}{g} + \frac{\pi^2\Lambda}{2g} \frac{|a-b|^2}{|\lambda|^2} \quad \text{for } \frac{|a-b|}{|\lambda|} \ll 1 \\ E &\simeq \frac{8\pi\Lambda}{g} + \frac{64\pi\Lambda}{g} \frac{|\lambda|^4}{|a-b|^4} \\ &\times \ln\left(\frac{|a-b|}{2|\lambda|}\right) \quad \text{for } \frac{|a-b|}{|\lambda|} \gg 1. \end{aligned} \quad (\text{A59})$$

9. Skyrmion-Antiskyrmion Kinetic Energy

We now move to determine the kinetic energy of the time dependent skyrmion-antiskyrmion configuration:

$$w = \frac{[z - a(t)][z^* - b^*(t)]}{\lambda^2(t)}. \quad (\text{A60})$$

we substitute

$$\begin{aligned} \partial_0 w \partial_0 w^* &= 4 \frac{|\dot{\lambda}|^2}{|\lambda|^2} |w|^2 + \frac{|\dot{a}|^2}{|\lambda|^4} |z-b|^2 + \frac{|\dot{b}|^2}{|\lambda|^4} |z-a|^2 + 2 \frac{\dot{\lambda} \dot{a}^*}{|\lambda|^2 \lambda^*} (z-b) w + 2 \frac{\dot{\lambda}^* \dot{a}}{|\lambda|^2 \lambda} (z^* - b^*) w^* \\ &+ 2 \frac{\dot{\lambda} \dot{b}}{|\lambda|^2 \lambda^*} (z^* - a^*) w + 2 \frac{\dot{\lambda}^* \dot{b}^*}{|\lambda|^2 \lambda} (z-a) w^* + \frac{\dot{a} \dot{b}}{|\lambda|^4} (z^* - a^*)(z^* - b^*) + \frac{\dot{a}^* \dot{b}^*}{|\lambda|^4} (z-a)(z-b) \end{aligned} \quad (\text{A61})$$

into the kinetic term of the sigma model Lagrangian

$$L = \frac{2\Lambda}{g} \int d^2x \frac{\partial_0 w \partial_0 w^* - 2 \partial_z w \partial_{z^*} w^* - 2 \partial_{z^*} w \partial_z w^*}{(1+|w|^2)^2} \quad (\text{A62})$$

to obtain

$$\begin{aligned} L &= \int d^2x \left(\frac{8\Lambda}{g} \frac{|\dot{\lambda}|^2}{|\lambda|^2} \frac{|w|^2}{(1+|w|^2)^2} + \frac{2\Lambda}{g} \frac{|\dot{a}|^2}{|\lambda|^4} \frac{|z-b|^2}{(1+|w|^2)^2} + \frac{2\Lambda}{g} \frac{|\dot{b}|^2}{|\lambda|^4} \frac{|z-a|^2}{(1+|w|^2)^2} \right. \\ &+ \frac{4\Lambda}{g} \frac{\dot{\lambda} \dot{a}^*}{|\lambda|^2 \lambda^*} \frac{(z-b)w}{(1+|w|^2)^2} + \frac{4\Lambda}{g} \frac{\dot{\lambda}^* \dot{a}}{|\lambda|^2 \lambda} \frac{(z^* - b^*)w^*}{(1+|w|^2)^2} + \frac{4\Lambda}{g} \frac{\dot{\lambda} \dot{b}}{|\lambda|^2 \lambda^*} \frac{(z^* - a^*)w}{(1+|w|^2)^2} \\ &\left. + \frac{4\Lambda}{g} \frac{\dot{\lambda}^* \dot{b}^*}{|\lambda|^2 \lambda} \frac{(z-a)w^*}{(1+|w|^2)^2} + \frac{2\Lambda}{g} \frac{\dot{a} \dot{b}}{|\lambda|^4} \frac{(z^* - a^*)(z^* - b^*)}{(1+|w|^2)^2} + \frac{2\Lambda}{g} \frac{\dot{a}^* \dot{b}^*}{|\lambda|^4} \frac{(z-a)(z-b)}{(1+|w|^2)^2} \right). \end{aligned} \quad (\text{A63})$$

We now evaluate term by term of the Lagrangian. The

first is

$$\begin{aligned} L_1 &= \frac{8|\dot{\lambda}|^2}{g|\lambda|^2} \int d^2x \frac{|w|^2}{(1+|w|^2)^2} \\ &= \frac{4|\dot{\lambda}|^2}{g} \int dz dz^* \frac{|z-\tilde{a}|^2 |z-\tilde{b}|^2}{(1+|z-\tilde{a}|^2 |z-\tilde{b}|^2)^2} \end{aligned} \quad (\text{A64})$$

where we made the conformal transformation $z \rightarrow \lambda z$, and defined $\tilde{a} = a/\lambda$ and $\tilde{b} = b/\lambda$. As in subsection A 6, we define

$$A \equiv \frac{\tilde{a} + \tilde{b}}{2}, \quad B \equiv \frac{\tilde{a} - \tilde{b}}{2} \quad (\text{A65})$$

and make the change of origin $z \rightarrow z + A$, to obtain

$$\begin{aligned} L_1 &= \frac{4\Lambda}{g} |\dot{\lambda}|^2 \int dz dz^* \frac{|z - B|^2 |z + B|^2}{(1 + |z - B|^2 |z + B|^2)^2} \\ &= \frac{2\Lambda}{g} |\dot{\lambda}|^2 \int dudu^* \frac{|u|^2}{(1 + |u|^2)^2 |u + B^2|} \\ &= \frac{4\Lambda}{g} |\dot{\lambda}|^2 \int \frac{r^3 dr d\theta}{(1 + r^2)^2 \sqrt{r^2 + |B|^4 + 2r|B|^2 \cos \theta}} \\ &= \frac{16\Lambda}{g} |\dot{\lambda}|^2 \int \frac{r^3 K\left(\frac{4r|B|^2}{(r+|B|^2)^2}\right) dr}{(1 + r^2)^2 (r + |B|^2)} \\ &= \frac{\Lambda}{4g} \frac{|\dot{\lambda}|^2}{|\lambda|^6} |a - b|^6 \int_0^\infty \frac{R^3 K\left(\frac{4R}{(R+1)^2}\right) dR}{(1 + |a - b|^4 R^2 / 16|\lambda|^4)^2 (R+1)} \\ &\equiv m_\lambda \frac{|\dot{\lambda}|^2}{2} \end{aligned} \quad (\text{A66})$$

where the second line follows from the variable change $u = z^2 - B^2$, and the third by going to polar coordinates in the complex u plane. $K(x)$ is the Jacobian elliptic

function²⁷. We have

$$\begin{aligned} L_1 &\simeq \frac{2\pi^2 \Lambda}{g} |\dot{\lambda}|^2 \quad \text{for } \frac{|a - b|}{|\lambda|} \ll 1 \\ L_1 &\simeq \frac{32\pi \Lambda}{g} \frac{|\dot{\lambda}|^2 |\lambda|^2}{|a - b|^2} \\ &\quad \times \ln\left(\frac{|a - b|^2}{4|\lambda|^2}\right) \quad \text{for } \frac{|a - b|}{|\lambda|} \gg 1. \end{aligned} \quad (\text{A67})$$

The second term of the Lagrangian (A63) is

$$\begin{aligned} L_2 &= \frac{2|\dot{a}|^2}{g|\lambda|^4} \int d^2x \frac{|z - b|^2}{(1 + |w|^2)^2} \\ &= \frac{|\dot{a}|^2}{g} \int dz dz^* \frac{|z - \tilde{b}|^2}{(1 + |z - \tilde{a}|^2 |z - \tilde{b}|^2)^2} \\ &= \frac{|\dot{a}|^2}{g} \int dz dz^* \frac{|z|^2}{(1 + |z - 2B|^2 |z|^2)^2} \\ &= \frac{2|\dot{a}|^2}{g} \int_0^\infty \int_0^{2\pi} \frac{r^3 dr d\theta}{[1 + r^2(r^2 + 4|B|^2 - 4r|B| \cos \theta)]^2} \\ &\equiv \frac{|\dot{a}|^2}{2} m_a \left(\frac{|a - b|}{|\lambda|} \right) \end{aligned} \quad (\text{A68})$$

where we made the shift $z \rightarrow z + A - B$ from the second to the third line. The mass for the a coordinate is defined through the integral

$$m_a \left(\frac{|a - b|}{|\lambda|} \right) = \frac{4}{g} \int_0^\infty \int_0^{2\pi} \frac{r^3 dr d\theta}{[1 + r^2(r^2 + 4|B|^2 - 4r|B| \cos \theta)]^2} \quad (\text{A69})$$

with $B = (a - b)/(2\lambda)$. For $|a - b| \ll |\lambda|$, we have

$$m_a = \frac{2\pi}{g}. \quad (\text{A70})$$

For $|a - b| \gg |\lambda|$ with $r = |B|R$, we have

$$m_a \left(\frac{|a - b|}{|\lambda|} \right) = \frac{4}{g|B|^4} \int_0^\infty \int_0^{2\pi} \frac{R^3 dR d\theta}{[(1/|B|^4) + R^4 + 4R^2 - 4R^3 \cos \theta]^2} \simeq \frac{2^9 \pi |\lambda|^8}{g|a - b|^8}. \quad (\text{A71})$$

Similarly the third term of the Lagrangian (A63) is

$$L_3 = \frac{|\dot{b}|^2}{2} m_b \left(\frac{|a - b|}{|\lambda|} \right) \quad (\text{A72})$$

with

$$m_b \left(\frac{|a - b|}{|\lambda|} \right) = m_a \left(\frac{|a - b|}{|\lambda|} \right). \quad (\text{A73})$$

The rest of the terms of the Lagrangian (A63) come out to be zero by rotational invariance in the complex z plane.

-
- ¹ R. B. Laughlin, Adv. Phys. **47**, 943 (1998); B. A. Bernevig, D. Giuliano and R. B. Laughlin, An. of Phys. **311**, 182 (2004).
- ² T. Senthil, A. Vishwanath, L. Balents, S. Sachdev and M. P. A. Fisher, Science **303**, 1490 (2004).
- ³ F. D. M. Haldane, Phys. Rev. Lett. **61**, 1029 (1988).
- ⁴ S. Sachdev, *Low Dimensional Quantum Field Theories for Condensed Matter Physicists*, Proceedings of the Trieste Summer School 1992 (World Scientific, Singapore, 1994).
- ⁵ S. Chakravarty, B. I. Halperin and D. R. Nelson, Phys. Rev. B **39**, 2344 (1989).
- ⁶ E. Fradkin and M. Stone, Phys. Rev. B **38**, 7215 (1988); X. G. Wen and A. Zee, Phys. Rev. Lett. **61**, 1025 (1988).
- ⁷ T. Dombre and N. Read, Phys. Rev. B **38**, 7181 (1988).
- ⁸ D. J. Gross, Nucl. Phys. B **132**, 439 (1978).
- ⁹ K. Wilson and J. Kogut, Phys. Rep. C **12**, 75 (1974).
- ¹⁰ A. M. Polyakov, Phys. Lett. B **59**, 79 (1975).
- ¹¹ E. Brézin and J. Zinn-Justin, Phys. Rev. Lett. **36**, 691 (1976).
- ¹² E. Brézin and J. Zinn-Justin, Phys. Rev. B **14**, 3110 (1976).
- ¹³ D. R. Nelson and R. A. Pelcovits, Phys. Rev. B, **16**, 2191 (1977).
- ¹⁴ E. Brézin, J. Zinn-Justin and J. C. Le Guillou, Phys. Rev. D **14**, 2615 (1976).
- ¹⁵ S. Weinberg, Phys. Rev. D **8**, 3497 (1973).
- ¹⁶ A. V. Chubukov, S. Sachdev and J. Ye, Phys. Rev. B **49**, 11919 (1994).
- ¹⁷ L. V. Osviannikov, Dokl. Akad. Nauk. SSSR **109**, 1112 (1956).
- ¹⁸ C. G. Callan, Phys. Rev. D **2**, 1541 (1970); K. Symanzik, Comm. Math. Phys. **18**, 227 (1970).
- ¹⁹ J. Goldstone and R. Jackiw, Phys. Rev. D **11**, 1486 (1975); J.-L. Gervais and B. Sakita, Phys. Rev. D **11**, 2943 (1975); E. Tomboulis, Phys. Rev. D **12**, 1678 (1975).
- ²⁰ A. A. Belavin and A. M. Polyakov, JETP Lett. **22**, 245 (1975).
- ²¹ T. Skyrme, Proc. Royal Soc. London A **260**, 127 (1961).
- ²² S. R. Coleman and R. Jackiw, Annals Phys. **67**, 552 (1971).
- ²³ J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Fourth Edition, Oxford Univ. Press, Oxford, UK (2002), Chapter 31, Section 4.
- ²⁴ M. Campostrini *et. al.*, Phys. Rev. B **65**, 144520 (2002).
- ²⁵ F. D. M. Haldane, Phys. Rev. Lett. **50**, 1153 (1983).
- ²⁶ J. Goldstone, Nuovo Cimento **19**, 154 (1961); Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961); J. Goldstone, A. Salam, and S. Weinberg Phys. Rev. **127**, 965 (1962).
- ²⁷ M. Abramowitz and I. E. Stegun, *Handbook of Mathematical Functions, With Formulas, Graphs, and Mathematical Tables*, Dover Publications, New York, NY (1974).
- ²⁸ V. A. Fatteev, I. V. Frolov and A. S. Schwarz, Nucl. Phys. B **154**, 1 (1979).
- ²⁹ A. M. Polyakov, *Gauge Fields and Strings*, Harwood Academic Publishers, Chur, Switzerland (1987).
- ³⁰ Z. Nazario and D. I. Santiago, arXiv:cond-mat/0606386 (2006).